CHAPTER 7

SYSTEMS OF PARTICLES

So far we have treated objects as if they were point particles, having mass but no size. This is really not such a serious restriction, because all points of an object in simple translational motion move in identical fashion, and it makes no difference whether we treat the object as a particle or as an extended body. For many objects in motion, however, this restriction is not valid. When an object rotates as it moves, for instance, or when its parts vibrate relative to one another, it would not be valid to treat the entire object as a single particle. Even in these more complicated cases, there is one point of the object whose motion under the influence of external forces can be analyzed as that of a simple particle. This point is called the center of mass. *In this chapter we describe how to find the center of mass of an object, and we show that Newton's laws can be used to describe the motion of the center of mass of a complex system.*

7-1 THE MOTION OF A COMPLEX OBJECT

Figure 7-1 shows the motion of a baton being tossed between two jugglers. At first glance, the motion looks very complicated, and it may not be obvious how to apply Newton's laws to analyze the motion. The baton is certainly not behaving like a particle (all parts of it do *not* move in the same way), and it is not apparent that *any* part of the baton is traveling the parabolic path that we expect for particle-like projectiles.

Our formulation of Newton's laws was based on particle behavior. Sometimes we are able to treat complex objects like particles if all parts of the object move in the same way. Since this is not true for the motion of the baton in Fig. 7-1, we must find a new way to analyze its motion.

It is clear that the baton is undergoing two kinds of motion simultaneously: the translational motion associated with a projectile and the rotational motion of a rigid body (which will be treated starting with the next chapter). These combined motions may appear complicated; however, if we fix our attention on a special point associated with the object, the analysis becomes simple once again. We can consider the motion of the baton to be a combination of a parabolic trajectory of that point (as if there were no rotational motion) plus a rotation about that point (as if there were no translational motion). That special point is called the *center of mass.* Figure 7-2 shows a time-exposure photograph of the motion of a baton with an indicator light marking the

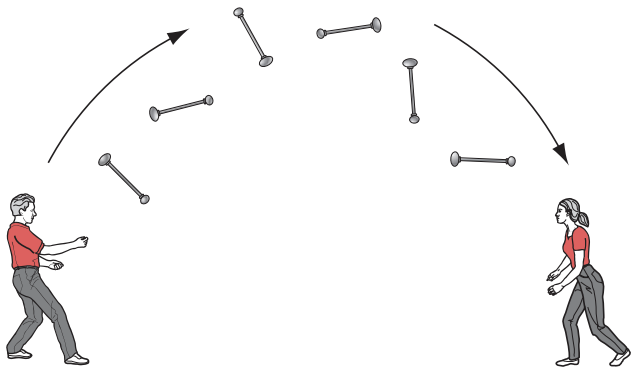

FIGURE 7-1. The complex motion of a baton as it is tossed between two jugglers.

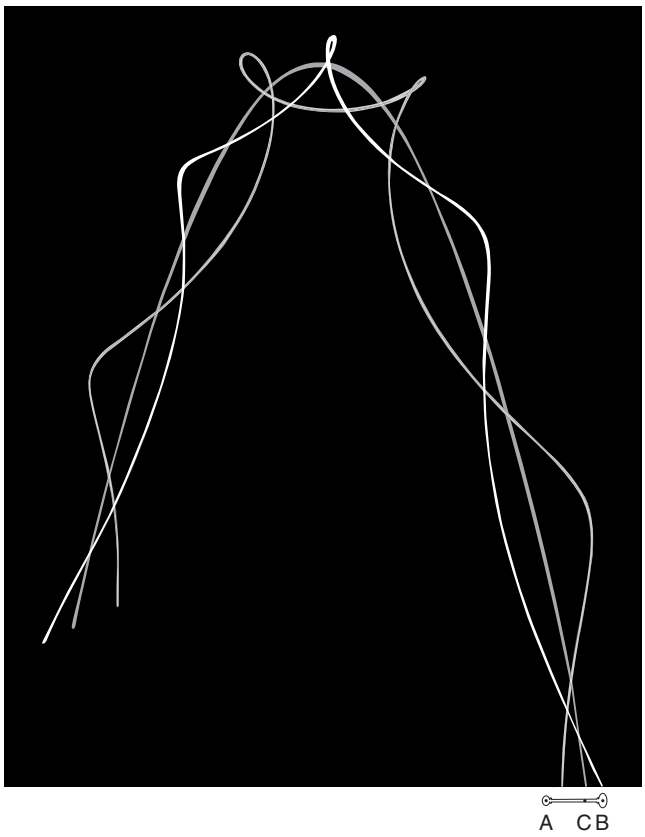

FIGURE 7-2. Time-exposure photograph of tossed baton with the motion of three points (*A*, *B*, *C*) indicated by lights. Points *A* and *B* show complex motions, but point *C* (the center of mass) follows a simple parabolic path. See "Center-of-Mass Baton" by Manfred Bucher *et al*., *The Physics Teacher,* February 1991, p. 74.

center of mass. This point moves in a simple parabolic trajectory, but no such simple description can be applied to the motion of other points of the baton.

Recall that in Section 6-5 we introduced the center of mass concept and found it useful in analyzing collisions between particles, but in that section we did not explain how to locate the center of mass of a system of particles. In this chapter we discuss how to find the center of mass of a solid object and how to use it to reduce a complex motion to a simpler one.

7-2 TWO-PARTICLE SYSTEMS

Let us try to simplify the problem discussed in the previous section. We consider the baton to consist of two particles, located at the ends *A* and *B*, connected by a thin rigid rod of fixed length and negligible mass. The mass of the particle at *B* is twice the mass of the particle at *A*.

We further simplify by sliding the baton on a frictionless horizontal surface instead of throwing it upward. This in effect eliminates gravity from the analysis.

In using Newton's laws to study the motion of a single object that we treated as a particle, we separated the problem into two parts: the particle and its environment. For a more complex object, such as our two-particle baton, it is usually more convenient to separate the problem into a *system* and its environment. The system can consist of any number of objects; we can define the system in any way that simplifies the problem, as long as we are consistent in the analysis and carefully account for all interactions between the system and its environment. These interactions are called *external forces.* Interactions between objects that are totally within the system are called *internal forces.* In the case of the baton, we define the system to consist of the two particles and the connecting rod; gravity and the normal force would then be classified as external forces, and the tension force exerted by the connecting rod on either of the particles would be an internal force.

We give the rod a push along the frictionless horizontal surface and examine its motion. Figure 7-3 shows a series of "snapshots" of the motions of the particles at *A* and *B* and the center of mass at *C*. Clearly each of the particles at *A* and *B* is accelerated, and so (according to Newton's second law) must be subject to a net force. However, point *C* shows no acceleration—its velocity is constant in both magnitude and direction. No other point associated with the baton moves in this simple way.

It is also interesting to view the motion of the baton from a frame of reference that is moving with the same velocity as point *C*. (As we shall see, this is the same as the center-of-mass reference frame we discussed in Section 6-5.) In this reference frame, point *C* would appear to be at rest. Figure 7-4 shows the resulting motion, with the positions of the baton drawn at the instants marked 1, 2, 3, and

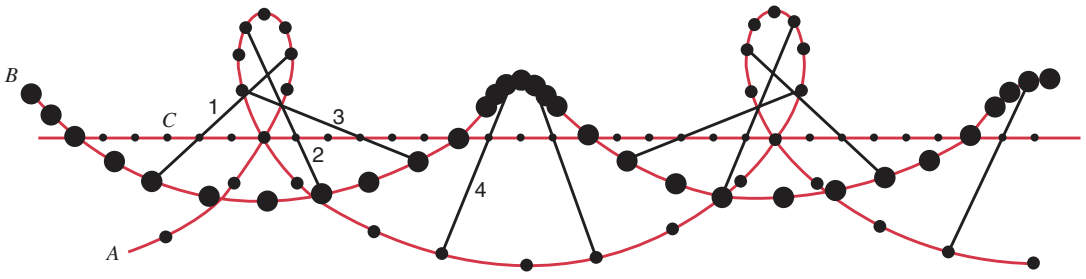

FIGURE 7-3. The motion of two particles attached to a connecting rod. The dots represent "snapshots" showing the locations of points *A*, *B*, and *C* at successive intervals of time. Point *C* on the rod follows a straight-line path and its successive positions are equally spaced, showing that it moves at constant velocity.

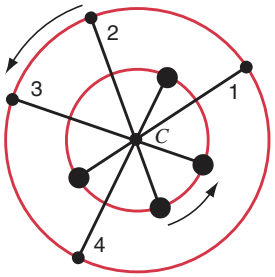

FIGURE 7-4. If we view the motion of Fig. 7-3 from a frame of reference moving along with point *C*, the rod appears to rotate about point *C* and the two particles move in circles of different radii.

4 in Fig. 7-3. The motion is a simple rotation with each particle having a constant rotational speed.

By focusing our attention on the center of mass *C*, we have been able to separate the complex motion of the system into two simple motions—the center of mass moves with constant velocity, and the system rotates with constant rotational speed about *C*. In the next chapter we shall consider the rotational motion; here we concentrate on the linear motion of the center of mass.

To find the location of the center of mass, we set up a coordinate system in the horizontal plane, as shown in Fig. 7-5. We let m_1 represent the mass of the particle at *A* and m_2 represent the particle at *B*. The vectors $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ locate m_1 and m_2 at a particular instant of time, relative to the origin we have chosen for the coordinate system. The center of mass is then located at that time by the vector $\vec{\mathbf{r}}_{cm}$:

$$\vec{\mathbf{r}}_{cm} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2} \tag{7-1}$$

or

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \quad \text{and} \quad y_{cm} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}. \tag{7-2}$$

The center of mass is a fixed point in any solid object whose location is determined by the way the mass of the object is distributed.

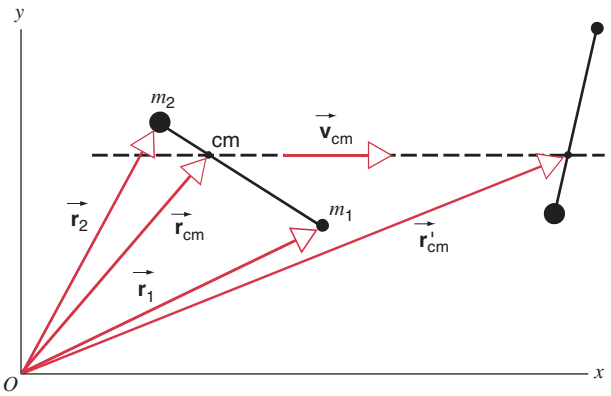

FIGURE 7-5. A coordinate system for locating the center of mass of our system at a particular time. At a later time, the center of mass is at $\vec{\mathbf{r}}'_{cm}$.

At a later time (as shown in Fig. 7-5), the system has moved to a new location and the center of mass has similarly changed location. To see why the motion of the center of mass is special, let us find its velocity and acceleration:

$$\vec{\mathbf{v}}_{cm} = \frac{d\vec{\mathbf{r}}_{cm}}{dt} = \frac{m_1\dfrac{d\vec{\mathbf{r}}_1}{dt} + m_2\dfrac{d\vec{\mathbf{r}}_2}{dt}}{m_1 + m_2} \tag{7-3}$$

or

$$\vec{\mathbf{v}}_{cm} = \frac{m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2}{m_1 + m_2}, \tag{7-4}$$

and

$$\vec{\mathbf{a}}_{cm} = \frac{d\vec{\mathbf{v}}_{cm}}{dt} = \frac{m_1\dfrac{d\vec{\mathbf{v}}_1}{dt} + m_2\dfrac{d\vec{\mathbf{v}}_2}{dt}}{m_1 + m_2} \tag{7-5}$$

or

$$\vec{\mathbf{a}}_{cm} = \frac{m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2}{m_1 + m_2}. \tag{7-6}$$

From Eq. 7-6 we can see why the motion of the center of mass of our system is so simple. Figure 7-6 shows free-body diagrams for the two particles and for the rod (assumed massless). For both particles, the vertical component of the acceleration is zero; thus the vertical component of the net force is zero, and the magnitudes of the normal force and weight are equal. The net force on m_1 is then $\vec{\mathbf{F}}_{1r}$ (the force on m_1 due to the connecting rod), and Newton's second law gives $\vec{\mathbf{F}}_{1r} = m_1\vec{\mathbf{a}}_1$. Similarly for m_2, $\vec{\mathbf{F}}_{2r} = m_2\vec{\mathbf{a}}_2$. By Newton's third law, the force on m_1 due to the connecting rod is equal and opposite to the force on the connecting rod by m_1, or $\vec{\mathbf{F}}_{1r} = -\vec{\mathbf{F}}_{r1}$; similarly, $\vec{\mathbf{F}}_{2r} = -\vec{\mathbf{F}}_{r2}$. Combining these results, the numerator of Eq. 7-6 becomes $m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 = \vec{\mathbf{F}}_{1r} + \vec{\mathbf{F}}_{2r} = -\vec{\mathbf{F}}_{r1} + (-\vec{\mathbf{F}}_{r2}) = -(\vec{\mathbf{F}}_{r1} + \vec{\mathbf{F}}_{r2})$. Finally, because we assumed that the connecting rod is massless, the net force on it must be zero ($\Sigma\,\vec{\mathbf{F}}_{rod} = m_{rod}\vec{\mathbf{a}}_{rod} = 0$ because $m_{rod} = 0$); see Fig. 7-6*b*. With $\Sigma\,\vec{\mathbf{F}}_{rod} = \vec{\mathbf{F}}_{r1} + \vec{\mathbf{F}}_{r2}$, the numerator of Eq. 7-6 becomes

$$m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 = -(\vec{\mathbf{F}}_{r1} + \vec{\mathbf{F}}_{r2}) = -\sum\vec{\mathbf{F}}_{rod} = 0.$$

We therefore have $\vec{\mathbf{a}}_{cm} = 0$, and so the center of mass moves with constant velocity.

In this discussion, we have so far assumed that no net external force acts on the system ($\vec{\mathbf{F}}_{1r}$ and $\vec{\mathbf{F}}_{2r}$ are *internal*

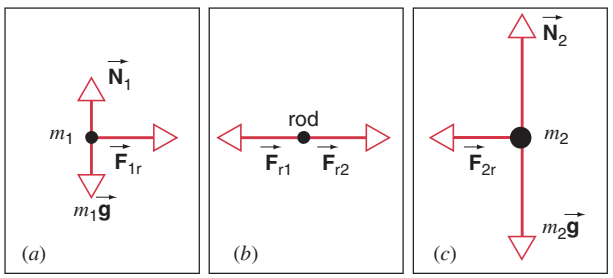

FIGURE 7-6. Free-body diagrams for (*a*) m_1, (*b*) the connecting rod, and (*c*) m_2.

forces, exerted by one part of the system on another part). Suppose now that there is an external force on each particle, perhaps a frictional force due to the surface. The net force on each particle is then the vector sum of the external force plus the internal force due to the rod:

$$\sum \vec{\mathbf{F}}_1 = \vec{\mathbf{F}}_{1,\text{ext}} + \vec{\mathbf{F}}_{1r} \quad \text{and} \quad \sum \vec{\mathbf{F}}_2 = \vec{\mathbf{F}}_{2,\text{ext}} + \vec{\mathbf{F}}_{2r}. \tag{7-7}$$

When we analyze Eq. 7-6 we then obtain

$$\begin{aligned} m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 &= \sum \vec{\mathbf{F}}_1 + \sum \vec{\mathbf{F}}_2 \\ &= \vec{\mathbf{F}}_{1,\text{ext}} + \vec{\mathbf{F}}_{1r} + \vec{\mathbf{F}}_{2,\text{ext}} + \vec{\mathbf{F}}_{2r}. \end{aligned} \tag{7-8}$$

Once again, $\vec{\mathbf{F}}_{1r} + \vec{\mathbf{F}}_{2r} = 0$, and defining the *net external force* to be $\Sigma \vec{\mathbf{F}}_{\text{ext}} = \vec{\mathbf{F}}_{1,\text{ext}} + \vec{\mathbf{F}}_{2,\text{ext}}$, Eq. 7-8 reduces to $m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 = \Sigma \vec{\mathbf{F}}_{\text{ext}}$. Using Eq. 7-6 we then obtain

$$\sum \vec{\mathbf{F}}_{\text{ext}} = (m_1 + m_2)\vec{\mathbf{a}}_{\text{cm}}. \tag{7-9}$$

This looks very much like Newton's second law, but it is applied to something that does not exist: a particle of mass $m_1 + m_2$ located at the center of mass.

Summarizing our conclusions for the two-particle system, we have seen that we can simplify our analysis if we decompose the complex motions into a motion of the center of mass of the system and another motion about the center of mass. If no net external force acts on the system, then the center of mass moves with constant velocity. If there is a net external force, then the motion of the center of mass can be found by assuming that the net external force acts on a particle located at the center of mass and having a mass equal to the total mass of the system. In the next section, we develop more general expressions that lead to the same conclusions for even more complex systems.

SAMPLE PROBLEM 7-1. (*a*) Suppose the baton of Fig. 7-3 is at rest along the x axis with the more massive particle $m_2 (= 2m)$ at coordinate x and the less massive particle $m_1 (= m)$ at coordinate $x + L$ (where L is the length of the rod connecting the particles), as shown in Fig. 7-7*a*. Find the center of mass. (*b*) Suppose now that m_2 is at the origin and the rod makes an angle of 45° between the x and y axes (Fig. 7-7*b*). Find the location of the center of mass.

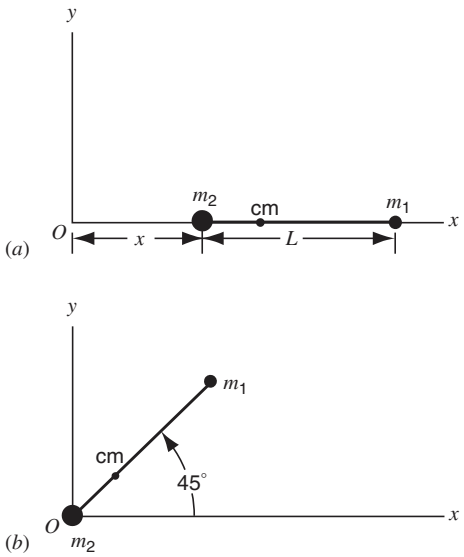

FIGURE 7-7. Sample Problem 7-1.

Solution (*a*) With $y_1 = 0$ and $y_2 = 0$, Eq. 7-2 gives $y_{\text{cm}} = 0$. The x coordinate of the center of mass is also found from Eq. 7-2:

$$x_{\text{cm}} = \frac{(m)(x + L) + (2m)(x)}{m + 2m} = x + \frac{L}{3}.$$

The center of mass is located along the rod a distance $L/3$ from the larger particle.

(*b*) In this case $x_1 = L/\sqrt{2}$, $y_1 = L/\sqrt{2}$, $x_2 = 0$, and $y_2 = 0$, and so we have

$$x_{\text{cm}} = \frac{(m)(L/\sqrt{2}) + (2m)(0)}{m + 2m} = \frac{L}{3\sqrt{2}},$$

$$y_{\text{cm}} = \frac{(m)(L/\sqrt{2}) + (2m)(0)}{m + 2m} = \frac{L}{3\sqrt{2}}.$$

Once again, the center of mass is along the connecting rod and $\frac{1}{3}$ of its length from the larger particle.

7-3 MANY-PARTICLE SYSTEMS

In this section we generalize the results of the previous section to systems in three dimensions that contain more than two particles.

We consider a system consisting of N particles of masses $m_1, m_2, \ldots, m_N$. The total mass is

$$M = m_1 + m_2 + \cdots + m_N = \sum m_n. \tag{7-10}$$

Each particle in the system can be represented by its mass m_n (where $n = 1, 2, \ldots, N$), its location at the coordinate $\vec{\mathbf{r}}_n$ (whose components are x_n, y_n, and z_n), its velocity $\vec{\mathbf{v}}_n$ (whose components are v_{nx}, v_{ny}, and v_{nz}), and its acceleration $\vec{\mathbf{a}}_n$. The net force on particle m_n is $\Sigma \vec{\mathbf{F}}_n$, which in general differs from one particle to another. This force may arise partly from the other $N - 1$ particles and partly from an external agent.

The center of mass of the system can be defined by a logical extension of Eq. 7-1:

$$\vec{\mathbf{r}}_{\text{cm}} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2 + \cdots + m_N\vec{\mathbf{r}}_N}{m_1 + m_2 + \cdots + m_N}$$

or

$$\vec{\mathbf{r}}_{\text{cm}} = \frac{1}{M}\sum m_n\vec{\mathbf{r}}_n. \tag{7-11}$$

In terms of components, the vector relationship of Eq. 7-11 can be written as

$$\begin{aligned} x_{\text{cm}} &= \frac{1}{M}(m_1x_1 + m_2x_2 + \cdots + m_Nx_N) \\ &= \frac{1}{M}\sum m_nx_n, \end{aligned} \tag{7-12a}$$

$$\begin{aligned} y_{\text{cm}} &= \frac{1}{M}(m_1y_1 + m_2y_2 + \cdots + m_Ny_N) \\ &= \frac{1}{M}\sum m_ny_n, \end{aligned} \tag{7-12b}$$

$$z_{cm} = \frac{1}{M}(m_1 z_1 + m_2 z_2 + \cdots + m_N z_N)$$
$$= \frac{1}{M}\sum m_n z_n. \qquad (7\text{-}12c)$$

Taking the derivative of Eq. 7-11, we find the velocity of the center of mass:

$$\vec{\mathbf{v}}_{cm} = \frac{d\vec{\mathbf{r}}_{cm}}{dt}$$
$$= \frac{1}{M}\left(m_1\frac{d\vec{\mathbf{r}}_1}{dt} + m_2\frac{d\vec{\mathbf{r}}_2}{dt} + \cdots + m_N\frac{d\vec{\mathbf{r}}_N}{dt}\right)$$

or

$$\vec{\mathbf{v}}_{cm} = \frac{1}{M}(m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 + \cdots + m_N\vec{\mathbf{v}}_N)$$
$$= \frac{1}{M}\sum m_n\vec{\mathbf{v}}_n. \qquad (7\text{-}13)$$

Differentiating once again, we find the acceleration of the center of mass:

$$\vec{\mathbf{a}}_{cm} = \frac{d\vec{\mathbf{v}}_{cm}}{dt} = \frac{1}{M}(m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 + \cdots + m_N\vec{\mathbf{a}}_N)$$
$$= \frac{1}{M}\sum m_n\vec{\mathbf{a}}_n. \qquad (7\text{-}14)$$

We can rewrite Eq. 7-14 as

$$M\vec{\mathbf{a}}_{cm} = m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2 + \cdots + m_N\vec{\mathbf{a}}_N$$

or

$$M\vec{\mathbf{a}}_{cm} = \sum\vec{\mathbf{F}}_1 + \sum\vec{\mathbf{F}}_2 + \cdots + \sum\vec{\mathbf{F}}_N, \qquad (7\text{-}15)$$

where the last result follows from applying Newton's second law, $\Sigma\,\vec{\mathbf{F}}_n = m_n\vec{\mathbf{a}}_n$, to each individual particle. The total force acting on a system of particles is thus equal to the total mass of the system times the acceleration of the center of mass. Equation 7-15 is just Newton's second law for the system of N particles treated as a single particle of mass M located at the center of mass, moving with velocity $\vec{\mathbf{v}}_{cm}$ and experiencing acceleration $\vec{\mathbf{a}}_{cm}$.

It is helpful to simplify Eq. 7-15 even a bit more. We can separate the force acting on each particle in the system into internal forces, which arise from the interactions with other particles that are part of the system, and external forces, which originate with the environment of the system under consideration. Any given particle m_n may experience force exerted on it by particle m_k, which we write as $\vec{\mathbf{F}}_{nk}$. This particular force is one among the many that make up $\Sigma\,\vec{\mathbf{F}}_n$, the total force on m_n. Similarly, the total force on particle m_k includes a term $\vec{\mathbf{F}}_{kn}$ due to the interaction with particle m_n. By Newton's third law, $\vec{\mathbf{F}}_{nk} = -\vec{\mathbf{F}}_{kn}$, and thus these two particular forces cancel when we carry out the sum of all the forces in Eq. 7-15. In fact, all such internal forces are part of action–reaction pairs and cancel. (In Chapter 3 we cautioned that the action and reaction forces must apply to different particles and thus do not oppose one another on a given particle. We are not violating that caution here, because we are applying the action to one particle and the reaction to another. The distinction here is that we are adding to get the net force on the *two* particles, in which case the action and reaction components, which still apply to different particles, do indeed cancel.)

All that remains in Eq. 7-15 is the total of all the *external* forces, and Eq. 7-15 reduces to

$$\sum\vec{\mathbf{F}}_{ext} = M\vec{\mathbf{a}}_{cm}, \qquad (7\text{-}16)$$

which can be written in terms of its components as

$$\sum F_{ext,x} = Ma_{cm,x}, \qquad \sum F_{ext,y} = Ma_{cm,y},$$

and

$$\sum F_{ext,z} = Ma_{cm,z}. \qquad (7\text{-}17)$$

We can summarize this important result as follows:

> *The overall translational motion of a system of particles can be analyzed using Newton's laws as if all the mass were concentrated at the center of mass and the total external force were applied at that point.*

A corollary follows immediately in the case $\Sigma\,\vec{\mathbf{F}}_{ext} = 0$:

> *If the net external force on a system of particles is zero, then the center of mass of the system moves with constant velocity.*

These are general results that apply equally well to collections of individual particles as they do to particles joined together by internal forces, as in a solid object. The object itself may be executing any sort of complicated motion, but the center of mass moves according to Eq. 7-16. Consider, for example, the motion of the baton of Fig. 7-1. As it travels, it also rotates. Its center of mass, however, follows a simple parabolic path. As far as the external force (gravity) is concerned, the system behaves as if it were a particle of mass M located at the center of mass. A complicated problem is therefore reduced to two relatively simple problems—the parabolic path of the center of mass and a rotation about the center of mass.

For another example, consider the Earth–Moon system moving under the Sun's gravity (the external force). Figure 7-8 shows that the center of mass of the system follows a

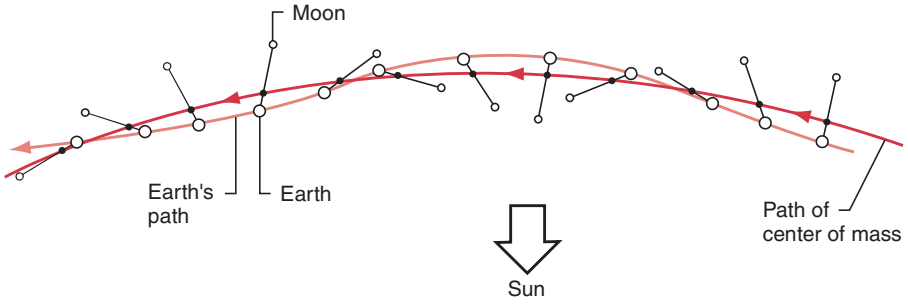

FIGURE 7-8. The center of mass of the Earth–Moon system follows a nearly circular orbit about the Sun, while the Earth and Moon rotate about their common center of mass, just like the baton of Fig. 7-3. This effect, which causes a slight "wobble" in the orbit of the Earth, is greatly exaggerated in the figure. The center of mass of the Earth–Moon system actually lies within the Earth, so the Earth always overlaps the orbital path of the center of mass.

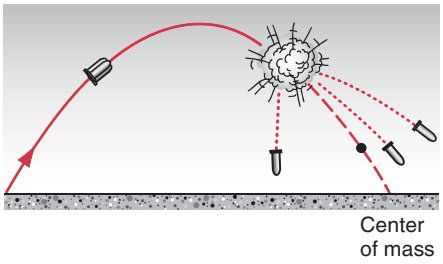

FIGURE 7-9. A projectile follows a parabolic path (solid line). An explosion breaks the projectile into three fragments, which travel so that their center of mass follows the original parabolic path.

stable orbit around the Sun; this is the path that would be followed by a particle of mass $m_{\text{Earth}} + m_{\text{Moon}}$. The Earth and Moon also rotate about their center of mass, resulting in a slight oscillation of the Earth about the path of the stable orbit. Using the data in Appendix C, you should be able to show that the center of mass of the Earth–Moon system lies within the Earth (see Exercise 1).

Figure 7-9 shows the motion of a projectile that breaks apart into three fragments. In effect, an explosion releases the three separate pieces, but since the explosion produces only internal forces it does not affect the motion of the center of mass. The center of mass continues to follow the parabolic path as if the explosion had not occurred, until one or more of the fragments experiences an external force, such as from atmospheric drag or impact with the ground.

SAMPLE PROBLEM 7-2. Figure 7-10*a* shows a system of three initially resting particles of masses $m_1 = 4.1$ kg, $m_2 = 8.2$ kg, and $m_3 = 4.1$ kg. The particles are each acted on by different net external forces, which have magnitudes $F_1 = 6$ N, $F_2 = 12$ N, and $F_3 = 14$ N. The directions of the forces are shown in the figure. Where is the center of mass of this system, and what is the acceleration of the center of mass?

Solution The position of the center of mass is marked by a dot in the figure. As Fig. 7-10*b* suggests, we treat this point as if it held a real particle, assigning to it a mass M equal to the system mass of $m_1 + m_2 + m_3 = 16.4$ kg and assuming that all external forces are applied at that point. We find the center of mass from Eqs. 7-12*a* and 7-12*b*:

$$x_{\text{cm}} = \frac{1}{M}(m_1x_1 + m_2x_2 + m_3x_3)$$

$$= \frac{1}{16.4\text{ kg}}[(4.1\text{ kg})(-2\text{ cm}) + (8.2\text{ kg})(4\text{ cm}) + (4.1\text{ kg})(1\text{ cm})] = 1.8\text{ cm},$$

$$y_{\text{cm}} = \frac{1}{M}(m_1y_1 + m_2y_2 + m_3y_3)$$

$$= \frac{1}{16.4\text{ kg}}[(4.1\text{ kg})(3\text{ cm}) + (8.2\text{ kg})(2\text{ cm}) + (4.1\text{ kg})(-2\text{ cm})] = 1.3\text{ cm}.$$

The x component of the net external force acting on the center of mass is (see Fig. 7-10*b*)

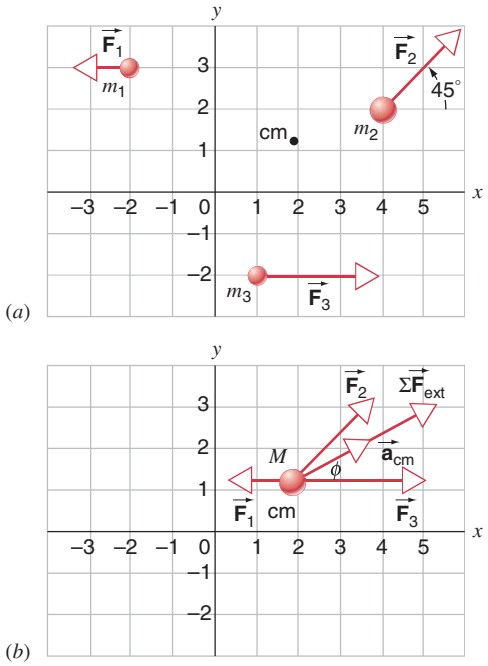

FIGURE 7-10. Sample Problem 7-2. (*a*) Three particles, placed at rest at the positions shown, are acted on by the forces shown. The center of mass of the system is marked. (*b*) The translational motion of the entire system can be represented by the motion of a particle with the total mass M located at the center of mass and acted on by the three external forces. The resultant force and the acceleration of the center of mass are shown. (The x and y axes are marked in centimeters.)

$$\sum F_{\text{ext},x} = F_{1x} + F_{2x} + F_{3x} = -6\text{ N} + (12\text{ N})(\cos 45°) + 14\text{ N} = 16.5\text{ N},$$

and the y component is

$$\sum F_{\text{ext},y} = F_{1y} + F_{2y} + F_{3y} = 0 + (12\text{ N})(\sin 45°) + 0 = 8.5\text{ N}.$$

The net external force thus has a magnitude of

$$\left|\sum \vec{\mathbf{F}}_{\text{ext}}\right| = \sqrt{(F_{\text{ext},x})^2 + (F_{\text{ext},y})^2} = \sqrt{(16.5\text{ N})^2 + (8.5\text{ N})^2} = 18.6\text{ N}$$

and makes an angle with the x axis given by

$$\phi = \tan^{-1}\frac{\sum F_{\text{ext},y}}{\sum F_{\text{ext},x}} = \tan^{-1}\frac{8.5\text{ N}}{16.5\text{ N}} = 27°.$$

This is also the direction of the acceleration vector. From Eq. 7-16, the magnitude of the acceleration of the center of mass is given by

$$|\vec{\mathbf{a}}_{\text{cm}}| = \frac{F_{\text{ext}}}{M} = \frac{18.6\text{ N}}{16.4\text{ kg}} = 1.1\text{ m/s}^2.$$

If the external forces are constant, then the acceleration of the center of mass is constant, even if the internal forces (and thus the accelerations of the individual particles) change with time.

SAMPLE PROBLEM 7-3. A projectile of mass 9.6 kg is launched from the ground with initial velocity of 12.4 m/s at an angle of 54° above the horizontal (Fig. 7-11). At some time after its launch, an explosion splits the projectile into two pieces. One piece, of mass 6.5 kg, is observed at 1.42 s after the launch at a height of 5.9 m and a horizontal distance of 13.6 m from the launch point. Find the location of the second fragment at that same time.

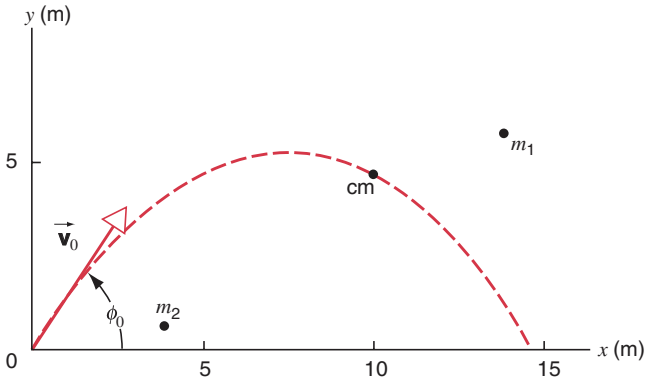

FIGURE 7-11. Sample Problem 7-3. The dashed line shows the parabolic trajectory of the center of mass of the two fragments. The locations of the center of mass and the two fragments are shown at $t = 1.42$ s.

Solution According to Eq. 7-16, the motion of the two fragments can be analyzed in terms of the motion of the combined system. Therefore, at $t = 1.42$ s after the launch, the center of mass of the two fragments must be at the same location where the original projectile would have been if it had not exploded. Let us first find that location. The location of the original projectile at $t = 1.42$ s can be found using Eqs. 4-10 with $v_{0x} = v_0 \cos \phi_0 = (12.4 \text{ m/s})\cos 54° = 7.3$ m/s and $v_{0y} = v_0 \sin \phi_0 = (12.4 \text{ m/s})\sin 54° = 10.0$ m/s. With the origin of the coordinate system at the original launch point, we have

$$x = v_{0x}t = (7.3 \text{ m/s})(1.42 \text{ s}) = 10.4 \text{ m},$$
$$y = v_{0y}t - \tfrac{1}{2}gt^2 = (10.0 \text{ m/s})(1.42 \text{ s}) - \tfrac{1}{2}(9.80 \text{ m/s}^2)(1.42 \text{ s})^2 = 4.3 \text{ m}$$

Because the motion of the center of mass of the system of the two fragments is the same as the motion of the original combined system would have been, the center of mass of the fragments at $t = 1.42$ s must be at $x_{cm} = 10.4$ m, $y_{cm} = 4.3$ m. We are given the location of one fragment, m_1, at that time: $x_1 = 13.6$ m, $y_1 = 5.9$ m. We can find the location of the other fragment, of mass $m_2 = M - m_1 = 9.6 \text{ kg} - 6.5 \text{ kg} = 3.1$ kg, by solving Eqs. 7-12*a* and 7-12*b* for x_2 and y_2:

$$x_2 = \frac{Mx_{cm} - m_1x_1}{m_2} = \frac{(9.6 \text{ kg})(10.4 \text{ m}) - (6.5 \text{ kg})(13.6 \text{ m})}{3.1 \text{ kg}} = 3.7 \text{ m},$$

$$y_2 = \frac{My_{cm} - m_1y_1}{m_2} = \frac{(9.6 \text{ kg})(4.3 \text{ m}) - (6.5 \text{ kg})(5.9 \text{ m})}{3.1 \text{ kg}} = 0.9 \text{ m}.$$

Figure 7-11 shows the location of the fragment m_2.

If we know the velocity of one fragment, we can use similar methods to find the velocity of the other (see Exercise 12).

In our analysis, we have assumed that gravity is the only external force that acts on the system, which allows us to represent the motion of the center of mass of the two fragments as the parabolic path of a projectile in the Earth's gravity. If one fragment struck the ground, there would be an additional force in the problem (the force of the ground on one fragment), and the center of mass would follow a different path. To use this method in that case, it would be necessary to know the force exerted by the ground.

7-4 CENTER OF MASS OF SOLID OBJECTS

It is far too tedious to find the center of mass of a solid object by using Eqs. 7-12 and summing over every atom in the system. Instead we divide the object into tiny elements of mass δm_n. As these elements become infinitesimally small, the sums of Eqs. 7-12 transform into integrals:

$$x_{cm} = \frac{1}{M}\lim_{\delta m \to 0}\sum x_n\,\delta m_n = \frac{1}{M}\int x\,dm, \qquad (7\text{-}18a)$$

$$y_{cm} = \frac{1}{M}\lim_{\delta m \to 0}\sum y_n\,\delta m_n = \frac{1}{M}\int y\,dm, \qquad (7\text{-}18b)$$

$$z_{cm} = \frac{1}{M}\lim_{\delta m \to 0}\sum z_n\,\delta m_n = \frac{1}{M}\int z\,dm. \qquad (7\text{-}18c)$$

In vector form (compare Eq. 7-11) these equations can be written

$$\vec{\mathbf{r}}_{cm} = \frac{1}{M}\int \vec{\mathbf{r}}\,dm. \qquad (7\text{-}19)$$

In many cases it is possible to use arguments based on geometry or symmetry to simplify the calculation of the center of mass of solid objects. If an object has *spherical symmetry,* the center of mass must lie at the geometrical center of the sphere. (It is not necessary that its density be uniform; a baseball, for example, has spherical symmetry even though it is composed of layers of different materials. Its center of mass is at its geometric center. When we refer to spherical symmetry, we mean that the density may vary with r but it must have the same variation in every direction.) If a solid has *cylindrical symmetry* (that is, if its mass is distributed symmetrically about an axis), then the center of mass must lie on the axis. If its mass is distributed symmetrically about a plane, then the center of mass must be in the plane.

Often we encounter solid, irregular objects that can be divided into several parts. We can find the center of mass of each part, and then by treating each part as a particle located at its own center of mass we can find the center of mass of the combination.

As an example, consider the triangular plate shown in Fig. 7-12. We divide the plate into a large number of thin strips parallel to the base of the triangle, as in Fig. 7-12*a*. The

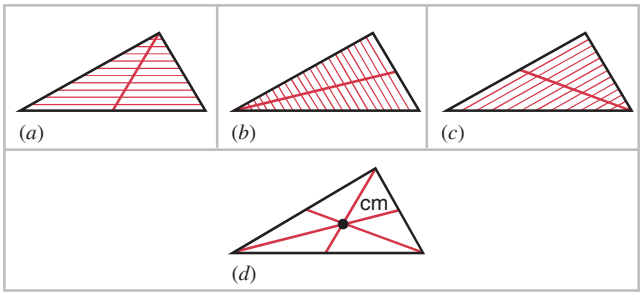

FIGURE 7-12. In (*a*), (*b*), and (*c*), the triangle is divided into thin strips, parallel to each of the three sides. The center of mass must lie along the symmetrical dividing lines shown. (*d*) The dot, the only point common to all three lines, is the position of the center of mass.

center of mass of each strip must lie at its geometrical center, and therefore the center of mass of the plate must lie somewhere along the line connecting the centers of the strips. (Replace each strip with a point mass located at the center of mass of the strip. The row of point masses forms in effect a one-dimensional object whose center of mass must surely lie along its length). Repeating this procedure for strips drawn parallel to the other two sides (Fig. 7-12*b* and 7-12*c*), we obtain two additional lines, each of which must also include the center of mass of the plate. Superimposing all three lines, as in Fig. 7-12*d*, we find they have only one point in common, which must therefore be the center of mass.

SAMPLE PROBLEM 7-4. Figure 7-13*a* shows a circular metal plate of radius $2R$ from which a disk of radius R has been removed. Let us call it object X. Its center of mass is shown as a dot on the x axis. Find the location of this point.

Solution Object X has symmetry about the x axis; that is, the portion above the x axis is the mirror image of the portion below the x axis. Because of this symmetry, the center of mass must lie along the x axis. Furthermore, because there is more of object X to the right of the y axis than to the left, the center of mass must lie to the right of the y axis. It is thus very reasonable that point X represents the center of mass of object X.

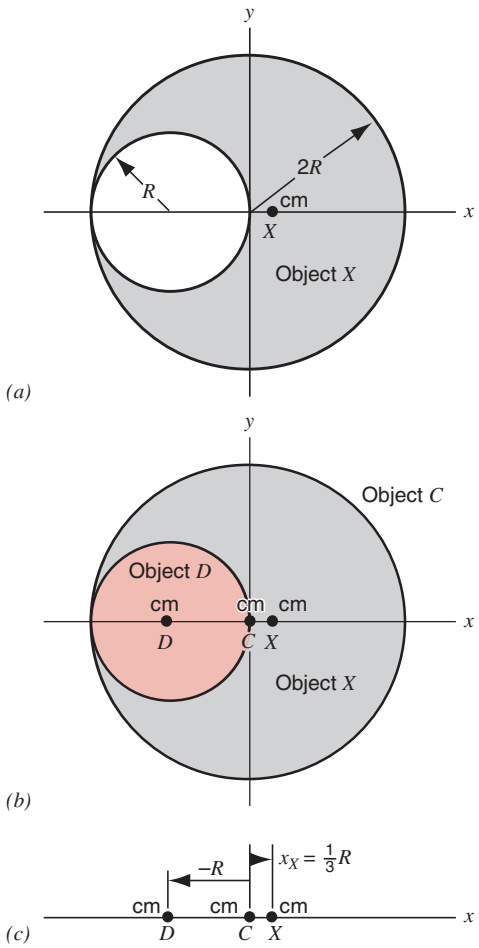

FIGURE 7-13. Sample Problem 7-4. (*a*) Object X is a metal disk of radius $2R$ with a hole of radius R cut in it. (*b*) Object D is a metal disk that fills the hole in object X; its center of mass is at $x_D = -R$. Object C is the composite disk made up of objects X and D; its center of mass is at the origin. (*c*) The centers of mass of the three objects.

Figure 7-13*b* shows object X, its hole filled with a disk of the same material of radius R, which we call object D. Let us label as object C the large uniform composite disk so formed. From symmetry, the center of mass of object C is at the origin of the coordinate system, as shown.

In finding the center of mass of a composite object, we can assume that the masses of its components are concentrated at their individual centers of mass. Thus object C can be treated as equivalent to two particles, representing objects X and D. Figure 7-13*c* shows the positions of the centers of mass of these three objects.

The position of the center of mass of object C is given from Eq. 7-12*a* as

$$x_C = \frac{m_D x_D + m_X x_X}{m_D + m_X},$$

in which x_D and x_X are the positions of the centers of mass of objects D and X, respectively. Noting that $x_C = 0$ and solving for x_X, we obtain

$$x_X = -\left(\frac{m_D}{m_X}\right) x_D.$$

The ratio m_D/m_X must be the same as the ratio of the areas of objects D and X (assuming the plate is of uniform density and thickness). That is,

$$\frac{m_D}{m_X} = \frac{\text{area of } D}{\text{area of } X} = \frac{\text{area of } D}{\text{area of } C - \text{area of } D}$$

$$= \frac{\pi R^2}{\pi(2R)^2 - \pi R^2} = \frac{1}{3}.$$

With $x_D = -R$, we obtain

$$x_X = \tfrac{1}{3}R.$$

SAMPLE PROBLEM 7-5. A thin strip of material is bent into the shape of a semicircle of radius R (Fig. 7-14). Find its center of mass.

Solution This case has symmetry about the y axis (that is, for every particle to the left of the y axis there is a particle in a similar loca-

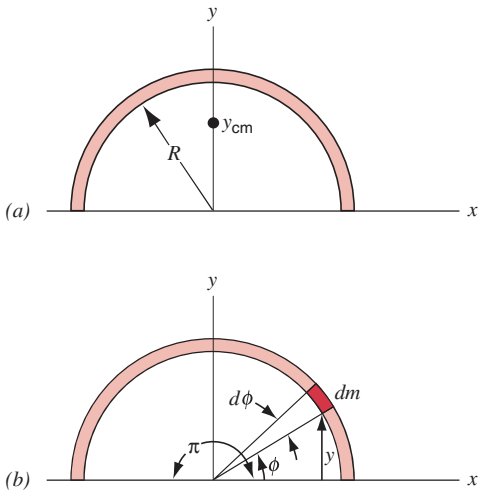

FIGURE 7-14. Sample Problem 7-5. (*a*) A thin strip of metal bent into the shape of a semicircle. (*b*) An element of the strip of mass dm located at the angular coordinate ϕ.

tion to the right of the y axis). The center of mass must therefore lie on the y axis; that is, $x_{cm} = 0$. However, there is no symmetry about the x axis, so we must use Eq. 7-18b to find y_{cm}. Using an angular coordinate simplifies the integration to be performed. Consider the small element of mass dm shown in Fig. 7-14b. It subtends an angle $d\phi$, and since the total mass M of the strip subtends an angle π (a full circle would subtend an angle 2π), the mass dm must be the same fraction of M as $d\phi$ is of π. That is, $dm/M = d\phi/\pi$, or $dm = (M/\pi)d\phi$. The element dm is located at the coordinate $y = R \sin \phi$. In this case we can write Eq. 7-18b as

$$y_{cm} = \frac{1}{M}\int y\, dm = \frac{1}{M}\int_0^{\pi} (R \sin \phi)\frac{M}{\pi}\, d\phi$$
$$= \frac{R}{\pi}\int_0^{\pi} \sin \phi\, d\phi = \frac{2R}{\pi} = 0.637R.$$

The center of mass is roughly two-thirds of a radius along the y axis. Note that, as this case illustrates, the center of mass does not need to be within the volume or the material of an object.

SAMPLE PROBLEM 7-6. A ball of mass m and radius R is placed inside a spherical shell of the same mass m and inner radius $2R$. The combination is at rest on a table top as shown in Fig. 7-15a. The ball is released, rolls back and forth inside, and finally comes to rest at the bottom, as in Fig. 7-15c. What will be the displacement d of the shell during this process?

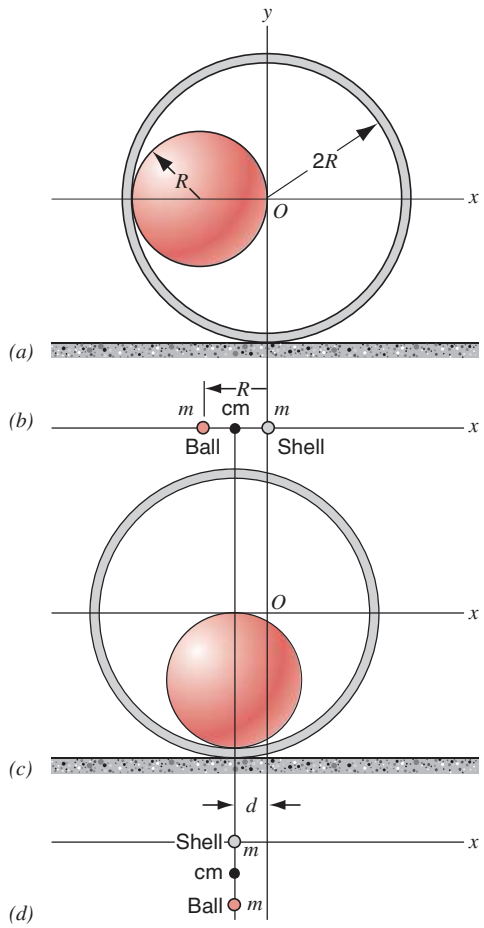

FIGURE 7-15. Sample Problem 7-6. (a) A ball of radius R is released from this initial position and is free to roll inside a spherical shell of radius $2R$. (b) The centers of mass of the ball, the shell, and their combination. (c) The final state after the ball has come to rest. The shell has moved so that the horizontal coordinate of the center of mass of the system remains in place. (d) The centers of mass of the ball, the shell, and their combination.

Solution The only external forces acting on the ball–shell system are the downward force of gravity and the normal force exerted vertically upward by the table. Neither force has a horizontal component so that $\Sigma F_{ext,x} = 0$. From Eq. 7-16 the acceleration component $a_{cm,x}$ of the center of mass must also be zero. Thus the horizontal position of the center of mass of the system must remain fixed, and the shell must move in such a way as to make sure that this happens.

We can represent both ball and shell by single particles of mass m, located at their respective centers. Figure 7-15b shows the system before the ball is released and Fig. 7-15d after the ball has come to rest at the bottom of the shell. We choose our origin to coincide with the initial position of the center of the shell. Figure 7-15b shows that, with respect to this origin, the center of mass of the ball–shell system is located a distance $\frac{1}{2}R$ to the left, halfway between the two particles. Figure 7-15d shows that the displacement of the shell is given by

$$d = \tfrac{1}{2}R.$$

The shell must move to the left through this distance as the ball comes to rest.

The ball is brought to rest by the frictional force that acts between it and the shell. Why does this frictional force not affect the final location of the center of mass?

7-5 CONSERVATION OF MOMENTUM IN A SYSTEM OF PARTICLES

Suppose we have a system containing N particles. The particles have masses m_n ($n = 1, 2, \ldots, N$) and move with velocities $\vec{\mathbf{v}}_n$ and momenta $\vec{\mathbf{p}}_n = m_n\vec{\mathbf{v}}_n$. The total momentum $\vec{\mathbf{P}}$ of the system is then

$$\vec{\mathbf{P}} = \sum_{n=1}^{N} \vec{\mathbf{p}}_n = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 + \cdots + \vec{\mathbf{p}}_N$$
$$= \sum_{n=1}^{N} m_n\vec{\mathbf{v}}_n = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 + \cdots + m_N\vec{\mathbf{v}}_N, \qquad (7\text{-}20)$$

which, according to Eq. 7-13, can be written as

$$\vec{\mathbf{P}} = M\vec{\mathbf{v}}_{cm}. \qquad (7\text{-}21)$$

Here $M = m_1 + m_2 + \cdots + m_N$ is the total mass of the system. Equation 7-21 gives us a different but equivalent definition of the total momentum of a system of particles:

The total momentum of a system of particles is equal to the product of the total mass of a system and the velocity of its center of mass.

Assuming a constant mass M, the derivative of the momentum is

$$\frac{d\vec{\mathbf{P}}}{dt} = M\frac{d\vec{\mathbf{v}}_{cm}}{dt} = M\vec{\mathbf{a}}_{cm}, \qquad (7\text{-}22)$$

using Eq. 7-14. Comparison of Eq. 7-22 with Eq. 7-16, $\Sigma \vec{\mathbf{F}}_{ext} = M\vec{\mathbf{a}}_{cm}$, allows us to write Newton's second law for a system of particles as

$$\sum \vec{\mathbf{F}}_{ext} = \frac{d\vec{\mathbf{P}}}{dt}. \tag{7-23}$$

Equation 7-23 states that, in a system of particles, the net external force equals the rate of change of the linear momentum of the system. This equation is the generalization of the single-particle expression $\Sigma \vec{\mathbf{F}} = d\vec{\mathbf{p}}/dt$, Eq. 6-2, to a system of many particles. Equation 7-23 reduces to Eq. 6-2 for the special case of a single particle, since only external forces can act on a one-particle system.

The law of conservation of linear momentum, which we derived in Section 6-4 for a system of two particles, also applies to a system of many particles, as we can see immediately from Eq. 7-23: If the net external force acting on a system is zero, then $d\vec{\mathbf{P}}/dt = 0$ and so the total linear momentum $\vec{\mathbf{P}}$ of the system remains constant.

If we view the system from a frame of reference that is moving with the center of mass, then in this frame the velocity $\vec{\mathbf{v}}'_n$ of a particle in the system is

$$\vec{\mathbf{v}}'_n = \vec{\mathbf{v}}_n - \vec{\mathbf{v}}_{cm}. \tag{7-24}$$

In this center-of-mass reference frame, the total momentum is, using Eq. 7-13,

$$\begin{aligned}\vec{\mathbf{P}}' &= \sum_{n=1}^{N} m_n\vec{\mathbf{v}}'_n = \sum_{n=1}^{N} m_n\vec{\mathbf{v}}_n - \sum_{n=1}^{N} m_n\vec{\mathbf{v}}_{cm} \\ &= M\vec{\mathbf{v}}_{cm} - \vec{\mathbf{v}}_{cm}\sum_{n=1}^{N} m_n \\ &= M\vec{\mathbf{v}}_{cm} - \vec{\mathbf{v}}_{cm}M = 0,\end{aligned} \tag{7-25}$$

so the center-of-mass reference frame is also the frame in which the total momentum is zero. This justifies our choice in Section 6-5 of the center-of-mass velocity (compare Eq. 6-23 with Eq. 7-4) for the reference frame from which to view the two-particle collision—only in this reference frame is the total momentum of the colliding particles zero both before and after the collision.

So far we are considering only systems in which the total mass M remains constant. Special care must be taken when applying Eq. 7-23 to systems in which the mass can change. The use of Eq. 7-23 in analyzing variable-mass systems is discussed in Section 7-6.

SAMPLE PROBLEM 7-7. A stream of bullets whose mass m is each 3.8 g is fired horizontally with a speed v of 1100 m/s into a large wooden block of mass M (= 12 kg) that is initially at rest on a horizontal table; see Fig. 7-16. If the block is free to slide without friction across the table, what speed will it acquire after it has absorbed eight bullets?

Solution For the moment we consider only the horizontal direction, which we define as the x axis, with positive to the right in Fig. 7-16. The x component of Eq. 7-23 is $\Sigma F_{ext,x} = dP_x/dt$. The net external force on the block has a nonzero horizontal component (due to the bullets), and the net external force on a bullet has a horizontal component (due to the block). However, if we choose our system to include both block and bullets, the forces between them are internal forces. No net horizontal external force acts on this system, and so the x component of the momentum must remain constant. We have identified the boundary of this system in Fig. 7-16. The initial (horizontal) momentum, measured while the bullets are still in flight and the block is at rest, is

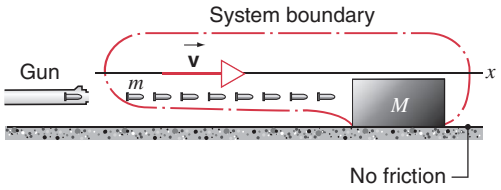

FIGURE 7-16. Sample Problem 7-7. A gun fires a stream of bullets toward a block of wood. We analyze the system that we define to be the block plus the bullets in flight.

$$P_{ix} = N(mv),$$

in which mv is the momentum of an individual bullet and $N = 8$. The final horizontal momentum, measured when all the bullets are in the block and the block is sliding over the table with horizontal velocity V, is

$$P_{fx} = (M + Nm)V.$$

Conservation of momentum requires that

$$P_{ix} = P_{fx}$$

or

$$N(mv) = (M + Nm)V.$$

Solving for V yields

$$V = \frac{Nm}{M + Nm}v = \frac{(8)(3.8 \times 10^{-3}\ \text{kg})}{12\ \text{kg} + (8)(3.8 \times 10^{-3}\ \text{kg})}(1100\ \text{m/s})$$

$$= 2.8\ \text{m/s}.$$

In the vertical direction, the external forces are the weight of the bullets, the weight of the block, and the normal force on the block. While the bullets are in flight, they acquire a small vertical momentum component as a result of the action of gravity. When the bullets strike the block, the block must exert on each bullet a force with both horizontal and vertical components. Along with the vertical force on the bullet, which is necessary to change its vertical momentum to zero, there must (according to Newton's third law) be a corresponding increase in the normal force exerted on the block by the horizontal surface. This increase is not only from the weight of the imbedded bullet; it has an additional contribution arising from the rate of change of the vertical momentum of the bullet. When all the bullets have come to rest relative to the block, the normal force will equal the combined weights of block and imbedded bullets.

For simplicity in solving this problem, we have assumed that the bullets are fired so rapidly that all eight are in flight before the first bullet strikes the block. Can you solve this problem without making this assumption?

Suppose the system boundary is enlarged so that it includes the gun, which is fixed to the Earth. Does the horizontal momentum of this system change before and after the firing? Is there a horizontal external force?

SAMPLE PROBLEM 7-8. As Fig. 7-17 shows, a cannon whose mass M is 1300 kg fires a 72-kg ball in a horizontal direction with a speed v of 55 m/s relative to the cannon. The cannon is mounted so that it can recoil freely. (*a*) What is the velocity V of the recoiling cannon with respect to the Earth? (*b*) What is the initial velocity v_E of the ball with respect to the Earth?

Solution (*a*) The coordinate system is set up with the positive x axis to the right in Fig. 7-17. Our inertial reference frame is fixed with respect to the Earth.

We choose the cannon plus the ball as our system. By doing so, the forces associated with the firing of the cannon are internal to the system, and we do not have to deal with them. The external forces acting on the system have no horizontal components. Thus the horizontal component of the total linear momentum of the system must remain unchanged as the cannon is fired.

In terms of vectors, $\vec{\mathbf{v}}_E = \vec{\mathbf{v}} + \vec{\mathbf{V}}$; that is, the velocity of the ball with respect to the Earth equals the velocity of the ball with respect to the cannon plus the velocity of the cannon with respect to the Earth. In the horizontal direction, we have $v_{Ex} = v_x + V_x$ where, as shown in the figure, we expect that the x component of $\vec{\mathbf{V}}$ is negative.

In the reference frame of the Earth, the horizontal component of the initial momentum P_{ix} is zero. After the cannon has fired, the final momentum of the system with respect to the Earth is that of the cannon ball plus the recoiling cannon:

$$P_{fx} = MV_x + mv_{Ex} = MV_x + m(v_x + V_x).$$

With $\Sigma F_{ext,x} = 0$, we must have $P_{ix} = P_{fx}$ and so

$$MV_x + m(v_x + V_x) = 0.$$

Solving for V_x gives

$$V_x = -\frac{mv_x}{M + m} = -\frac{(72\text{ kg})(55\text{ m/s})}{1300\text{ kg} + 72\text{ kg}} = -2.9\text{ m/s}.$$

The negative sign tells us that the cannon recoils to the left in Fig. 7-17, as we expect it should.

(*b*) With respect to the Earth, the horizontal component of the velocity of the ball is

$$v_{Ex} = v_x + V_x = 55\text{ m/s} + (-2.9\text{ m/s}) = 52\text{ m/s}.$$

Because of the recoil, the ball is moving a little slower with respect to the Earth than it otherwise would. Note the importance in this problem of choosing the system (cannon + ball) wisely and being absolutely clear about the reference frame (Earth or recoiling cannon) to which the various measurements are referred.

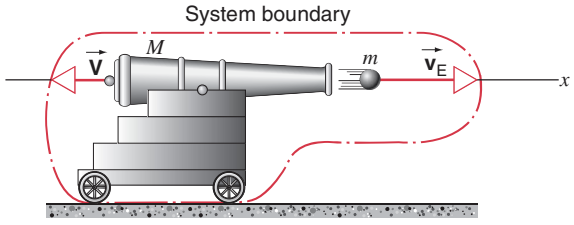

FIGURE 7-17. Sample Problem 7-8. A cannon of mass M fires a ball of mass m. The velocities of the ball and the recoiling cannon are shown in a reference frame fixed with respect to the Earth. Velocities are taken as positive to the right.

7-6 SYSTEMS OF VARIABLE MASS* (Optional)

Imagine that the cart holding the cannon of Fig. 7-17 also holds a large stack of cannonballs. As the cannon is repeatedly fired, the cart (which we assume to move without friction) recoils to the left, and with each recoil its speed increases. With the system boundary drawn as in Fig. 7-17, we know that the total horizontal momentum must be zero and that there is no net horizontal force on the system. If, however, we consider a system including only the cannon plus cart, then the previous statement is no longer true. The momentum of the cannon increases each time it is fired, and it is appropriate for us to use the familiar language of Newtonian physics to account for the change in momentum through the action of a suitable force. In this case, the force that accelerates the cannon is a reaction force: the cannon, by virtue of its exploding charge, pushes on the cannonballs to eject them, and the reaction force (the cannonballs pushing back on the cannon) moves the cannon to the left.

As the cannon is repeatedly fired, the total mass on the cart decreases by the quantity of cannonballs that have been ejected. The methods of Sample Problem 7-8 cannot easily be used to solve this problem, because the mass M of the recoiling object is different every time the cannon fires.

We refer to the system S consisting of cannon plus cart (including the unfired cannon balls) in this example as a "variable-mass" system. Of course, the larger system S' consisting of cannon plus all fired cannonballs is a constant-mass system *and* a constant-momentum system (in the absence of external force). The smaller system S, however, does not have constant mass. Moreover, the ejected cannonballs carry momentum, and there is a net outflow of momentum from S that is responsible for its acceleration.

The above example gives a reasonably good mental image of how a rocket works. Fuel is burned and ejected at high speed; the combustion products correspond to the cannonballs. The rocket (less the consumed fuel) experiences an acceleration that depends on the rate at which fuel is consumed and the speed with which it is ejected.

The goal in analyzing systems similar to the rocket is *not* to consider the kinematics of the entire system S'. Instead, we focus our attention on one particular subsystem S, and we ask how S moves as the mass within the entire system S' is redistributed so that the mass within subsystem S changes. The total mass within S' remains constant, but the particular subsystem S we consider can change its state of motion as it gains or loses mass (and momentum).

Figure 7-18 shows a schematic view of a generalized system. At time t, the subsystem S has a mass M and moves with velocity $\vec{\mathbf{v}}$ in the particular inertial frame of reference from which we are observing. At a time $t + \Delta t$, the mass of

* See "Force, Momentum Change, and Motion," by Martin S. Tiersten, *American Journal of Physics,* January 1969, p. 82, for an excellent general reference on systems of fixed and variable mass.

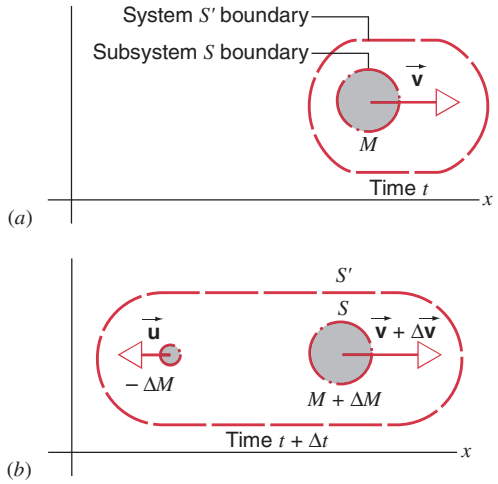

FIGURE 7-18. (*a*) A system S' at time t consists of a mass M moving with velocity $\vec{\mathbf{v}}$. (*b*) At a time Δt later, the original mass M has ejected some mass $-\Delta M$. The remaining mass $M + \Delta M$, which we call the subsystem S, now moves with velocity $\vec{\mathbf{v}} + \Delta\vec{\mathbf{v}}$.

S has changed by an amount ΔM (a negative quantity, in the case of ejected mass) to $M + \Delta M$, while the mass of the remainder of the full system S' has changed by a corresponding amount $-\Delta M$. The system S now moves with a velocity $\vec{\mathbf{v}} + \Delta\vec{\mathbf{v}}$, and the ejected matter moves with velocity $\vec{\mathbf{u}}$, both measured from our frame of reference.

To make the situation as general as possible, we also allow for an external force that may act on the entire system. In the case of a rocket, this is not the force that propels the rocket (which is an internal force for the system S'), but instead is the force due to some external agent, perhaps gravity or atmospheric drag. The total momentum of the entire system S' is $\vec{\mathbf{P}}$, and Newton's second law can be written

$$\sum \vec{\mathbf{F}}_{\text{ext}} = \frac{d\vec{\mathbf{P}}}{dt}. \tag{7-26}$$

In the time interval Δt, the change in momentum $\Delta\vec{\mathbf{P}}$ is

$$\Delta\vec{\mathbf{P}} = \vec{\mathbf{P}}_{\text{f}} - \vec{\mathbf{P}}_{\text{i}}, \tag{7-27}$$

where $\vec{\mathbf{P}}_{\text{f}}$, the final momentum of the system S' at time $t + \Delta t$, and $\vec{\mathbf{P}}_{\text{i}}$, the initial momentum of S' at time t, are given by

$$\vec{\mathbf{P}}_{\text{i}} = M\vec{\mathbf{v}}, \tag{7-28a}$$

$$\vec{\mathbf{P}}_{\text{f}} = (M + \Delta M)(\vec{\mathbf{v}} + \Delta\vec{\mathbf{v}}) + (-\Delta M)\vec{\mathbf{u}}. \tag{7-28b}$$

The change in momentum of S' is thus

$$\begin{aligned}\Delta\vec{\mathbf{P}} &= \vec{\mathbf{P}}_{\text{f}} - \vec{\mathbf{P}}_{\text{i}} \\ &= (M + \Delta M)(\vec{\mathbf{v}} + \Delta\vec{\mathbf{v}}) + (-\Delta M)\vec{\mathbf{u}} - M\vec{\mathbf{v}}.\end{aligned} \tag{7-29}$$

Rewriting the derivative in Eq. 7-26 as a limit and substituting this expression for $\Delta\vec{\mathbf{P}}$, we obtain

$$\begin{aligned}\sum \vec{\mathbf{F}}_{\text{ext}} &= \lim_{\Delta t \to 0} \frac{\Delta\vec{\mathbf{P}}}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{(M + \Delta M)(\vec{\mathbf{v}} + \Delta\vec{\mathbf{v}}) + (-\Delta M)\vec{\mathbf{u}} - M\vec{\mathbf{v}}}{\Delta t} \\ &= \lim_{\Delta t \to 0} \left[M\frac{\Delta\vec{\mathbf{v}}}{\Delta t} + (\vec{\mathbf{v}} - \vec{\mathbf{u}})\frac{\Delta M}{\Delta t} + \Delta\vec{\mathbf{v}}\frac{\Delta M}{\Delta t} \right] \\ &= M\frac{d\vec{\mathbf{v}}}{dt} + (\vec{\mathbf{v}} - \vec{\mathbf{u}})\frac{dM}{dt}.\end{aligned} \tag{7-30}$$

Note that, in taking the limit, the last term in the square brackets vanishes, because $\Delta\vec{\mathbf{v}} \to 0$ as $\Delta t \to 0$. In Eq. 7-30, M is the mass of the subsystem S at time t, and $d\vec{\mathbf{v}}/dt$ is its acceleration as it gains or loses mass at velocity $\vec{\mathbf{u}}$ (in our frame of reference) and at a rate $|dM/dt|$. If $dM/dt > 0$, the mass of the subsystem increases; if $dM/dt < 0$, its mass decreases.

We can write Eq. 7-30 in a more instructive form as

$$M\frac{d\vec{\mathbf{v}}}{dt} = \sum \vec{\mathbf{F}}_{\text{ext}} + \vec{\mathbf{v}}_{\text{rel}}\frac{dM}{dt}, \tag{7-31}$$

where $\vec{\mathbf{v}}_{\text{rel}} = \vec{\mathbf{u}} - \vec{\mathbf{v}}$ is the velocity of the gained or lost matter relative to the subsystem S. For example, if subsystem S is a rocket, $\vec{\mathbf{v}}_{\text{rel}}$ is the velocity of the ejected gases *relative to the rocket.* This is a reasonable quantity to introduce, because the speed of the ejected gases is a fundamental design characteristic of a rocket engine and should not be expressed in a form that depends on any frame of reference other than the rocket itself.

Equation 7-31 shows that the acceleration $d\vec{\mathbf{v}}/dt$ of the subsystem S (the rocket, for example) is in part determined by the net external force and in part by the momentum transferred by the mass that is gained or lost. Note that $\vec{\mathbf{v}}_{\text{rel}} = \vec{\mathbf{u}} - \vec{\mathbf{v}}$ points to the left in Fig. 7-18; since dM/dt is negative for a rocket, the second term in Eq. 7-31 is represented by a vector that points to the right and so is responsible for accelerating the subsystem in that direction. This term is called the *thrust* of the rocket and can be interpreted as the force exerted on the rocket by the ejected gas. The thrust of a rocket can be increased by increasing either the speed of the ejected gas or the rate at which it is ejected.

If mass is ejected at a constant rate and at a constant speed relative to S, the thrust is constant but the acceleration is not constant, because M is decreasing. If $dM/dt = 0$, so that the mass of the subsystem does not change, Eq. 7-31 reduces to our familiar form of Newton's second law, $\Sigma\,\vec{\mathbf{F}}_{\text{ext}} = M\vec{\mathbf{a}}$.

The analogy between a rocket and a recoiling gun is apparent from Fig. 7-19. In each case momentum is conserved for the entire system, consisting of the ejected mass (bullets or fuel) plus the object that ejects the mass. When we focus our attention on the gun or the rocket within the larger system, we see that its mass changes and that there is a force that drives it, a recoil in the case of the gun and a thrust in the case of the rocket. If we view the system from a reference frame at the center of mass, then as time passes there is more ejected mass, and it has traveled further to the

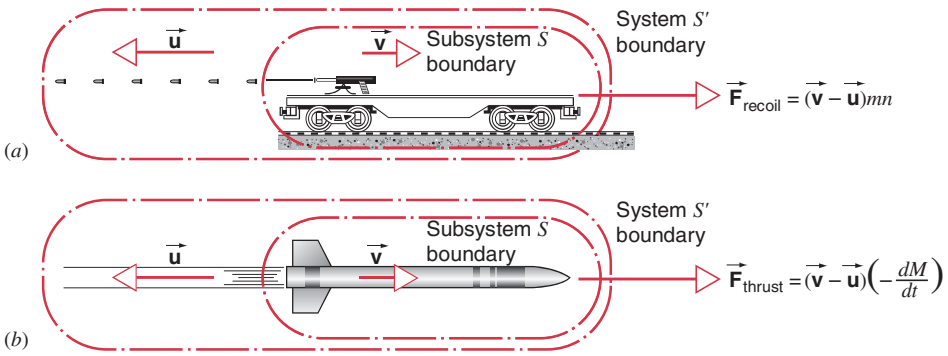

FIGURE 7-19. (*a*) A machine gun fires a stream of bullets at a rate of n per unit time. The total momentum of the system S' remains constant, but the subsystem S experiences a recoil force that changes its momentum. Its change in momentum in a time dt is exactly equal to the opposite momentum $mn\vec{\mathbf{u}}\,dt$ carried by the bullets. (*b*) A rocket ejects a stream of combustion products. The total momentum of the system S' remains constant, but the subsystem S experiences a thrust that changes its momentum. Its change in momentum in a time dt is exactly equal to the opposite momentum $\vec{\mathbf{u}}\,dM$ carried by the ejected gas.

left in Fig. 7-19, meaning that the object must travel to the right to keep the center of mass fixed.

SAMPLE PROBLEM 7-9. A spaceship with a total mass of 13,600 kg is moving relative to a certain inertial reference frame with a speed of 960 m/s in a region of space of negligible gravity. It fires its rocket engines to give an acceleration parallel to the initial velocity. The rockets eject gas at a constant rate of 146 kg/s with a constant speed (relative to the spaceship) of 1520 m/s, and they are fired until 9100 kg of fuel has been burned and ejected. (*a*) What is the thrust produced by the rockets? (*b*) What is the velocity of the spaceship after the rockets have fired?

Solution (*a*) The thrust is given by the last term of Eq. 7-31. Its magnitude is

$$F = \left| v_{\text{rel}} \frac{dM}{dt} \right| = (1520 \text{ m/s})(146 \text{ kg/s}) = 2.22 \times 10^5 \text{ N}.$$

(*b*) Choosing the positive x direction to be that of the spaceship's initial velocity, we can write Eq. 7-31 (with $\Sigma \vec{\mathbf{F}}_{\text{ext}} = 0$) as

$$M \frac{dv_x}{dt} = v_{\text{rel},x} \frac{dM}{dt}.$$

Because the gas is ejected relative to the rocket in a direction opposite to its velocity (which we have chosen as our positive x direction), $v_{\text{rel},x}$ is negative. Since dM/dt is also negative, the right side of this equation is positive, and the spaceship's velocity increases. Rewriting the equation as $dv_x = v_{\text{rel},x}\,(dM/M)$, we can integrate on the left from the initial velocity of 960 m/s to the final velocity we seek to determine. On the right we integrate from the initial mass (13,600 kg) to the final mass (13,600 kg − 9100 kg = 4500 kg)

$$\int_{v_{ix}}^{v_{fx}} dv_x = v_{\text{rel},x} \int_{M_i}^{M_f} \frac{dM}{M},$$

which we can evaluate as

$$v_{fx} - v_{ix} = v_{\text{rel},x} \ln \frac{M_f}{M_i}. \tag{7-32}$$

Solving for v_{fx} we find

$$v_{fx} = 960 \text{ m/s} + (-1520 \text{ m/s}) \ln \frac{4500 \text{ kg}}{13{,}600 \text{ kg}} = 2640 \text{ m/s}.$$

SAMPLE PROBLEM 7-10. Sand drops from a stationary hopper at a rate of 0.134 kg/s onto a conveyor belt moving with a speed of 0.96 m/s, as shown in Fig. 7-20. What net force must be applied to the conveyor belt to keep it moving at constant speed?

Solution We choose the direction of motion of the belt as our positive x direction, and we fix our coordinate system in the laboratory that locates our inertial reference frame (and in which the hopper is at rest). The system S' includes the belt and all of the sand in the hopper. The subsystem S represents the belt and only the sand that has dropped onto it. The mass of S is increasing ($dM/dt > 0$) as more sand drops onto the belt.

We can apply Eq. 7-31 to this situation, with $dv_x/dt = 0$ (because the belt moves with constant velocity) and also with $u_x = 0$

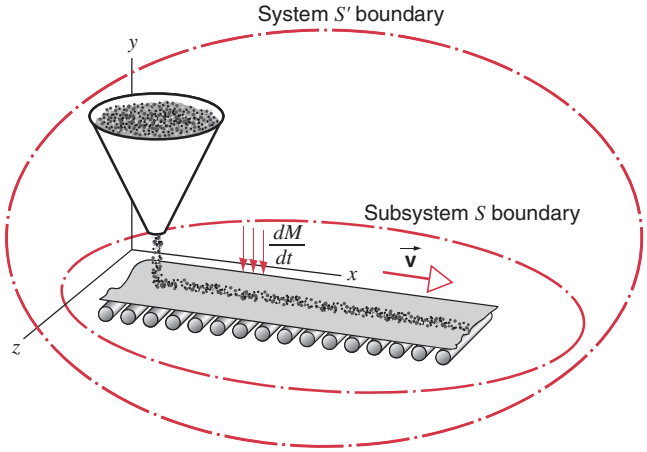

FIGURE 7-20. Sample Problem 7-10. Sand drops from a hopper at a rate dM/dt onto a conveyor belt moving with constant velocity $\vec{\mathbf{v}}$ in the reference frame of the laboratory. The hopper is at rest in the reference frame of the laboratory.

(because the sand dropping onto the belt has no component of velocity in the x direction. Thus $v_{\text{rel},x} = -v_x$; that is, an observer traveling with the belt would see the sand leaving the hopper (and the hopper itself) moving in the negative x direction. Solving for the net external force, we find

$$\sum F_{\text{ext},x} = -v_{\text{rel},x}\frac{dM}{dt} = v_x\frac{dM}{dt} = (0.96\text{ m/s})(0.134\text{ kg/s})$$
$$= 0.129\text{ N}.$$

The force has a positive x component; that is, it must be applied in the direction of motion of the belt to increase the x component of the velocity of each grain of sand that drops onto the belt from 0 to 0.96 m/s.

MULTIPLE CHOICE

7-1 The Motion of a Complex Object

7-2 Two-Particle Systems

1. Two frictionless pucks are connected by a rubber band. One of the pucks is projected across an air table, the rubber band tightens, and the second puck follows—in an apparently random way—the first puck. The center of mass of this two-particle system is located
(A) at a fixed distance from one of the pucks.
(B) usually, but not always, between the two pucks.
(C) at a distance from one of the pucks that is a fixed ratio to the distance between the two pucks.
(D) sometimes closer to one puck, and sometimes closer to the other.

2. Two objects are moving on a surface. The center of mass exists only if
(A) the two objects are physically connected.
(B) the surface is level.
(C) the surface is frictionless.
(D) There is always a center of mass.

3. Two objects are sitting on a level, frictionless surface. The objects are not connected or touching. A force F is applied to *one* of the objects, which then moves with acceleration a. Which of the following statements is most correct?
(A) The center of mass concept cannot be applied because the external force does not act on both objects.
(B) The center of mass moves with acceleration that could be greater than a.
(C) The center of mass moves with acceleration that must be equal to a.
(D) The center of mass moves with acceleration that must be less than a.

4. Two objects of unequal mass are connected by a light string that passes over a pulley. One of the objects is given an initial upward velocity. The center of mass of the two objects will
(A) accelerate up or down, depending on the relative masses of the two objects.
(B) accelerate downward only after it has reached some highest point.
(C) accelerate downward at some value less than g.
(D) accelerate downward with a value of g.

5. Two objects of unequal mass are connected by a compressed spring. The combined object is thrown vertically into the air. At the highest point of the trajectory the spring releases, resulting in one of the objects being projected even higher into the air; the spring remains attached to the other object. Shortly after the spring releases, the center of mass of the objects is
(A) moving upward and accelerating upward.
(B) moving upward but accelerating downward.
(C) moving downward but accelerating upward.
(D) moving downward and accelerating downward.
(E) There is not enough information given to answer the question.

7-3 Many-Particle Systems

6. Three objects are on a table. You can find the center of mass by
(A) combining the three objects together according to Eq. 7-12.
(B) combining the two lightest objects first according to Eq. 7-12, calling it a new "particle," and then combining the third object to this new particle.
(C) combining the two heaviest objects first, calling it a new "particle," and then adding the third object to this new particle.
(D) any of the above methods; they are equivalent.

7. Seven identical geese are flying south together at constant speed. A hunter shoots one of them, which immediately dies and falls to the ground. The other six continue flying south at the original speed. After the one goose has hit the ground, the center of mass of all seven geese
(A) continues south at the original speed, but is now located some distance behind the flying geese.
(B) continues south, but at $\frac{6}{7}$ the original speed.
(C) continues south, but at $\frac{1}{7}$ the original speed.
(D) stops with the dead goose.

8. Measure the height that a person can jump by how far above the ground her head rises. Can a person jump higher with her hands fixed above her head or with her hands fixed at her sides?
(A) With her hands fixed above head.
(B) With hands fixed at sides.
(C) The result is the same in both cases.
(D) The answer depends on the relative size of the person's hands compared to her overall mass.

7-4 Center of Mass of Solid Objects

9. A solid body has a center of mass located inside the body. A hole is drilled somewhere in the body, but not near the center of mass. After the hole has been drilled, the center of mass of the remaining body moves from the original position

(A) away from the hole. (B) toward the hole.
(C) not at all. (D) in a random direction.

10. Consider Sample Problem 7-6. The inner ball is released and rolls around inside the spherical shell for some time before coming to rest. During this time the center of mass
(A) moves both horizontally and vertically.
(B) moves both horizontally and vertically, but returns to the original horizontal position when the system comes to rest.
(C) moves only vertically.
(D) moves vertically only downward, because the center of mass can never move upward.

7-5 Conservation of Momentum in a System of Particles

11. A system of N particles is free from any external forces.
(*a*) Which of the following is true for the magnitude of the total momentum of the system?
(A) It must be zero.
(B) It could be non-zero, but it must be constant.
(C) It could be non-zero, and it might not be constant.
(D) The answer depends on the nature of the internal forces in the system.
(*b*) Which of the following must be true for the sum of the *magnitudes* of the momenta of the individual particles in the system?
(A) It must be zero.
(B) It could be non-zero, but it must be constant.
(C) It could be non-zero, and it might not be constant.
(D) It could be zero, even if the magnitude of the total momentum is not zero.

12. An isolated rail car of mass M is moving along a straight, frictionless track at an initial speed v_0. The car is passing under a bridge when a crate filled with N bowling balls, each of mass m, is dropped from the bridge into the bed of the rail car. The crate splits open and the bowling balls bounce around inside the rail car, but none of them fall out.
(*a*) Is the momentum of the rail car + bowling balls system conserved in this collision?
(A) Yes, the momentum is completely conserved.
(B) Only the momentum component in the vertical direction is conserved.
(C) Only the momentum component parallel to the track is conserved.
(D) No components are conserved.
(*b*) What is the average speed of the rail car + bowling balls system some time after the collision?
(A) $(M + Nm)v_0/M$ (B) $Mv_0/(Nm + M)$
(C) Nmv_0/M
(D) The speed cannot be determined because there is not enough information.

13. An isolated rail car originally moving with speed v_0 on a straight, frictionless, level track contains a large amount of sand. A release valve on the bottom of the car malfunctions, and sand begins to pour out *straight down* relative to the rail car.
(*a*) Is momentum conserved in this process?
(A) The momentum of the rail car alone is conserved.
(B) The momentum of the rail car + sand remaining within the car is conserved.
(C) The momentum of the rail car + *all* of the sand, both inside and outside the rail car, is conserved.
(D) None of the three previous systems have momentum conservation.
(*b*) What happens to the speed of the rail car as the sand pours out?
(A) The car begins to roll faster.
(B) The car maintains the same speed.
(C) The car begins to slow down.
(D) The problem cannot be solved since momentum is not conserved.

7-6 Systems of Variable Mass

QUESTIONS

1. A canoeist in a still pond can reach shore by jerking sharply on the rope attached to the bow of the canoe. How do you explain this? (It really can be done.)

2. How might a person sitting at rest on a frictionless horizontal surface get altogether off it?

3. A box is sitting on a frictionless, level surface. A small spring cannon, which is capable of firing a lump of clay, is lowered into the box on a string attached to the ceiling. Before the cannon is fired, the center of mass of the cannon, clay, and box is fixed at a point A. The cannon fires; the clay shoots from the cannon and sticks to the wall of the box. Does the center of mass of the cannon, clay, and box system move? Explain.

4. Does the center of mass of a solid object necessarily lie within the object? If not, give examples.

5. Figure 7-21 shows (*a*) an isosceles triangular prism and (*b*) a right circular cone whose diameter is the same length as the base of the triangle. The center of mass of the triangle is one-third of the way up from the base but that of the cone is only one-fourth of the way up. Can you explain the difference?

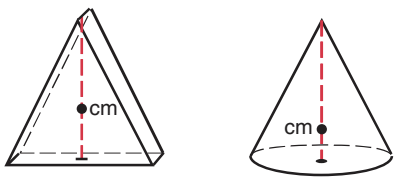

FIGURE 7-21. Question 5.

6. How is the center of mass concept related to the concept of geographic center of the country? To the population center of the country? What can you conclude from the fact that the geographic center differs from the population center?

7. Where is the center of mass of the Earth's atmosphere?

8. The center of mass of a full can of soda is approximately at the center. (*a*) Part of the soda is consumed and the can is replaced on the table. What happens to the center of mass of the can + soda system? (*b*) Eventually all of the soda is consumed and the can is replaced on the table. What happens to the center of mass of the can? See Problem 5.

9. An amateur sculptor decides to portray a bird (Fig. 7-22). Luckily, the final model is actually able to stand upright. The model is formed of a single thick sheet of metal of uniform thickness. Of the points shown, which is most likely to be the center of mass?

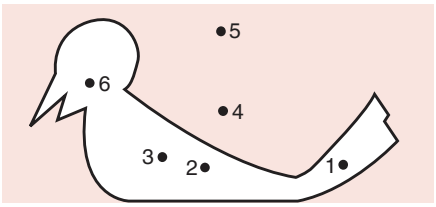

FIGURE 7-22. Question 9.

10. Someone claims that when a skillful high jumper clears the bar, the jumper's center of mass actually goes under the bar. Is this possible?

11. A ballet dancer doing a grand jete (great leap; see Fig. 7-23) seems to float horizontally in the central portion of her leap. Show how the dancer can maneuver her legs in flight so that, although her center of mass does indeed follow the expected parabolic trajectory, the top of her head moves more or less horizontally. (See "The Physics of Dance," by Kenneth Laws, *Physics Today,* February 1985, p. 24.)

FIGURE 7-23. Question 11.

12. Can a sailboat be propelled by air blown at the sails from a fan attached to the boat? Explain your answer.

13. If only an external force can change the state of motion of the center of mass of a body, how does it happen that the internal force of the brakes can bring a car to rest?

14. A man stands still on a large sheet of slick ice; in his hand he holds a lighted firecracker. He throws the firecracker at an angle (that is, not vertically) into the air. Describe briefly, but as exactly as you can, the motion of the center of mass of the firecracker and the motion of the center of mass of the system consisting of man and firecracker. It will be most convenient to describe each motion during each of the following periods: (*a*) after he throws the firecracker, but before it explodes; (*b*) between the explosion and the first piece of firecracker hitting the ice; (*c*) between the first fragment hitting the ice and the last fragment landing; and (*d*) during the time when all fragments have landed but none has reached the edge of the ice.

15. You throw an ice cube with velocity $\vec{\mathbf{v}}$ into a hot, gravity-free, evacuated space. The cube gradually melts to liquid water and then boils to water vapor. (*a*) Is it a system of particles all the time? (*b*) If so, is it the same system of particles? (*c*) Does the motion of the center of mass undergo any abrupt changes? (*d*) Does the total linear momentum change?

16. An evacuated box is at rest on a frictionless table. You punch a small hole in one face so that air can enter. (See Fig. 7-24.) How will the box move? What argument did you use to arrive at your answer?

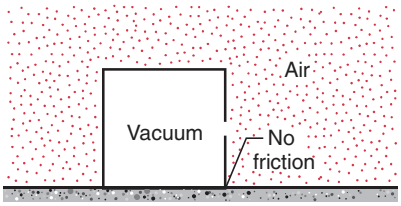

FIGURE 7-24. Question 16.

17. A railroad flatcar is initially at rest. It holds N people each of weight w. If each person in succession runs parallel to the tracks with a relative velocity v_{rel} and jumps off the end, do they impart to the car a greater velocity than if they all run and jump at the same time?

18. Can you think of variable-mass systems other than the examples given in the text?

19. It is not correct to use the equation $\Sigma \vec{\mathbf{F}}_{ext} = d(M\vec{\mathbf{v}})/dt$ for a system of variable mass. To show this (*a*) put the equation in the form $(\Sigma \vec{\mathbf{F}}_{ext} - M\, d\vec{\mathbf{v}}/dt)/(dM/dt) = \vec{\mathbf{v}}$ and (*b*) show that one side of this equation has the same value in all inertial frames, where as the other side does not. Hence the equation cannot be generally valid. (*c*) Show that Eq. 7-31 leads to no such contradiction.

20. In 1920 a prominent newspaper editorialized as follows about the pioneering rocket experiments of Robert H. Goddard, dismissing the notion that a rocket could operate in a vacuum: "That Professor Goddard, with his 'chair' in Clark College and the countenancing of the Smithsonian Institution, does not know the relation of action to reaction, and of the need to have something better than a vacuum against which to react—to say that would be absurd. Of course, he seems only to lack the knowledge ladled out daily in high schools." What is wrong with this argument?

21. The final velocity of the final stage of a multi-stage rocket is much greater than the final velocity of a single-stage rocket of the same total mass and fuel supply. Explain this fact.

22. Can a rocket reach a speed greater than the speed of the exhaust gases that propel it? Explain why or why not.

23. Are there any possible methods of propulsion in outer space other than rockets? If so, what are they and why are they not used?

24. Equation 7-32 suggests that the speed of a rocket can increase without limit if enough fuel is burned. Is this reasonable? What is the limit of applicability of Eq. 7-32? Where in our derivation of Eq. 7-32 did we introduce this limit? (See "The Equation of Motion for Relativistic Particles and Systems with Variable Rest Mass," by Kalman B. Pomeranz, *American Journal of Physics,* December 1964, p. 955.)

EXERCISES

7-1 The Motion of a Complex Object

7-2 Two-Particle Systems

1. How far is the center of mass of the Earth–Moon system from the center of the Earth? (From Appendix C, obtain the masses of the Earth and Moon and the distance between the centers of the Earth and Moon. It is interesting to compare the answer to the Earth's radius.)

2. Show that the ratio of the distances x_1 and x_2 of two particles from their center of mass is the inverse ratio of their masses; that is, $x_1/x_2 = m_2/m_1$.

3. A Plymouth with a mass of 2210 kg is moving along a straight stretch of road at 105 km/h. It is followed by a Ford with mass 2080 kg moving at 43.5 km/h. How fast is the center of mass of the two cars moving?

4. Two skaters, one with mass 65 kg and the other with mass 42 kg, stand on an ice rink holding a pole with a length of 9.7 m and a mass that is negligible. Starting from the ends of the pole, the skaters pull themselves along the pole until they meet. How far will the 42-kg skater move?

5. Two particles P and Q are initially at rest 1.64 m apart. P has a mass of 1.43 kg and Q a mass of 4.29 kg. P and Q attract each other with a constant force of 1.79×10^{-2} N. No external forces act on the system. (*a*) Describe the motion of the center of mass. (*b*) At what distance from P's original position do the particles collide?

6. A shell is fired from a gun with a muzzle velocity of 466 m/s, at an angle of 57.4° with the horizontal. At the top of the trajectory, the shell explodes into two fragments of equal mass. One fragment, whose speed immediately after the explosion is zero, falls vertically. How far from the gun does the other fragment land, assuming level terrain?

7. A dog weighing 10.8 lb is standing on a flatboat so that he is 21.4 ft from the shore. He walks 8.50 ft on the boat toward shore and then halts. The boat weighs 46.4 lb, and one can assume there is no friction between it and the water. How far is he from the shore at the end of this time? (Hint: The center of mass of boat + dog does not move. Why?) The shoreline is also to the left in Fig. 7-25.

FIGURE 7-25. Exercise 7.

8. Richard, mass 78.4 kg, and Judy, who is less massive, are enjoying Lake George at dusk in a 31.6-kg canoe. When the canoe is at rest in the placid water, they change seats, which are 2.93 m apart and symmetrically located with respect to the canoe's center. Richard notices that the canoe moved 41.2 cm relative to a submerged log and calculates Judy's mass. What is it?

9. An 84.4-kg man is standing at the rear of a 425-kg iceboat that is moving at 4.16 m/s across ice that may be considered to be frictionless. He decides to walk to the front of the 18.2-m-long boat and does so at a speed of 2.08 m/s with respect to the boat. How far does the boat move across the ice while he is walking?

7-3 Many-Particle Systems

10. Where is the center of mass of the three particles shown in Fig. 7-26?

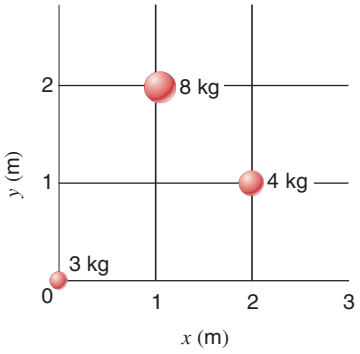

FIGURE 7-26. Exercise 10.

11. In the ammonia (NH_3) molecule, the three hydrogen (H) atoms form an equilateral triangle, the distance between centers of the atoms being 16.28×10^{-11} m, so that the center of the triangle is 9.40×10^{-11} m from each hydrogen atom. The nitrogen (N) atom is at the apex of a pyramid, the three hydrogens constituting the base (see Fig. 7-27). The nitrogen/hydrogen distance is 10.14×10^{-11} m and the nitro-

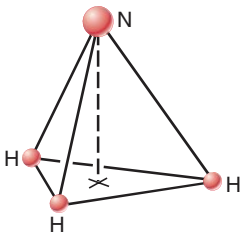

FIGURE 7-27. Exercise 11.

gen/hydrogen atomic mass ratio is 13.9. Locate the center of mass relative to the nitrogen atom.

12. Consider Sample Problem 7-3. The 6.5-kg fragment is observed at $t = 1.42$ s to be moving with a velocity whose horizontal component is 11.4 m/s in the same direction as that of the launch of the original projectile and whose vertical component is 4.6 m/s downward. Find the velocity of the 3.1-kg fragment at that time.

7-4 Center of Mass of Solid Objects

13. Three thin rods each of length L are arranged in an inverted U, as shown in Fig. 7-28. The two rods on the arms of the U each have mass M; the third rod has mass $3M$. Where is the center of mass of the assembly?

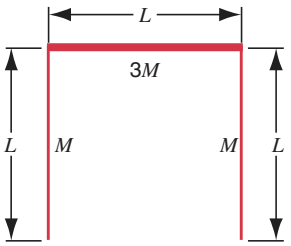

FIGURE 7-28. Exercise 13.

14. Fig. 7-29 shows a composite slab with dimensions 22.0 cm × 13.0 cm × 2.80 cm. Half of the slab is made of aluminum (density = 2.70 g/cm^3) and half of iron (density = 7.85 g/cm^3), as shown. Where is the center of mass of the slab?

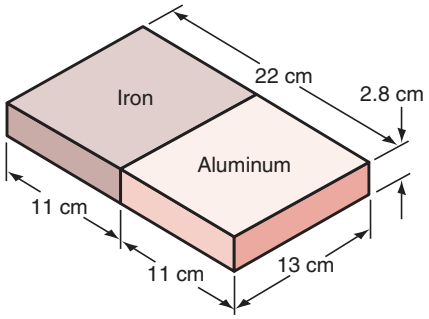

FIGURE 7-29. Exercise 14.

15. A box, open at the top, in the form of a cube of edge length 40 cm, is constructed from thin metal plate. Find the coordinates of the center of mass of the box with respect to the coordinate system shown in Fig. 7-30.

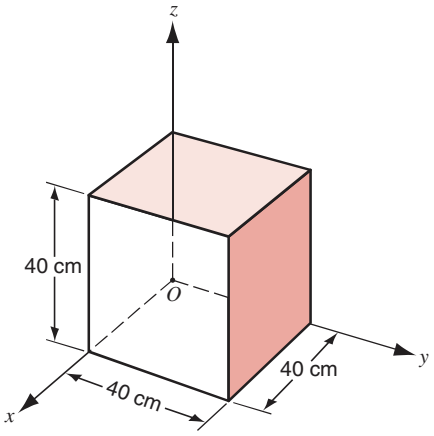

FIGURE 7-30. Exercise 15.

7-5 Conservation of Momentum in a System of Particles

16. A vessel at rest explodes, breaking into three pieces. Two pieces, one with twice the mass of the other, fly off perpendicular to one another with the same speed of 31.4 m/s. The third piece has three times the mass of the lightest piece. Find the magnitude and direction of its velocity immediately after the explosion. (Specify the direction by giving the angle from the line of travel of the least massive piece.)

17. Each minute, a special game warden's machine gun fires 220, 12.6-g rubber bullets with a muzzle velocity of 975 m/s. How many bullets must be fired at an 84.7-kg animal charging toward the warden at 3.87 m/s in order to stop the animal in its tracks? (Assume that the bullets travel horizontally and drop to the ground after striking the target.)

18. A railway flat car is rushing along a level frictionless track at a speed of 45 m/s. Mounted on the car and aimed forward is a cannon that fires 65-kg cannon balls with a muzzle speed of 625 m/s. The total mass of the car, the cannon, and the large supply of cannon balls on the car is 3500 kg. How many cannon balls must be fired to bring the car as close to rest as possible?

19. Twelve 100.0-kg containers of rocket parts in empty space are loosely tethered by ropes tied together at a common point. The center of mass of the twelve containers is originally at rest. A 50-kg lump of "space-goo" moving at 80 m/s collides with one of the containers and sticks to it. (*a*) Assuming that none of the tethers break, find the speed of the center of mass of the twelve containers after the collision with the space goo. (*b*) Assuming instead that the tether of the struck container does break, find the speed of the center of mass of the twelve containers after the collision.

7-6 Systems of Variable Mass

20. A rocket at rest in space, where there is virtually no gravity, has a mass of 2.55×10^5 kg, of which 1.81×10^5 kg is fuel. The engine consumes fuel at the rate of 480 kg/s, and the exhaust speed is 3.27 km/s. The engine is fired for 250 s. (*a*) Find the thrust of the rocket engine. (*b*) What is the mass of the rocket after the engine burn? (*c*) What is the final speed attained?

21. Consider a rocket at rest in empty space. What must be its mass ratio (ratio of initial to final mass) in order that, after firing its engine, the rocket's speed is (*a*) equal to the exhaust speed and (*b*) equal to twice the exhaust speed?

22. During a lunar mission, it is necessary to make a midcourse correction of 22.6 m/s in the speed of the spacecraft, which is moving at 388 m/s. The exhaust speed of the rocket engine is 1230 m/s. What fraction of the initial mass of the spacecraft must be discarded as exhaust?

23. A rocket of total mass 1.11×10^5 kg, of which 8.70×10^4 kg is fuel, is to be launched vertically. The fuel will be burned at the constant rate of 820 kg/s. Relative to the rocket, what is the minimum exhaust speed that allows liftoff at launch?

24. A 5.4-kg toboggan carrying 35 kg of sand slides from rest down an icy slope 93 m long, inclined 26° below the horizontal. The sand leaks from the back of the toboggan at the rate of 2.3 kg/s. How long does it take the toboggan to reach the bottom of the slope?

25. A freight car, open at the top, weighing 9.75 metric tons, is coasting along a level track with negligible friction at

1.36 m/s when it begins to rain hard. The raindrops fall vertically with respect to the ground. What is the speed of the car when it has collected 0.50 metric tons of rain? What assumptions, if any, must you make to get your answer?

PROBLEMS

1. A man of mass m clings to a rope ladder suspended below a balloon of mass M; see Fig. 7-31. The balloon is stationary with respect to the ground. (*a*) If the man begins to climb the ladder at a speed v (with respect to the ladder), in what direction and with what speed (with respect to the Earth) will the balloon move? (*b*) What is the state of motion after the man stops climbing?

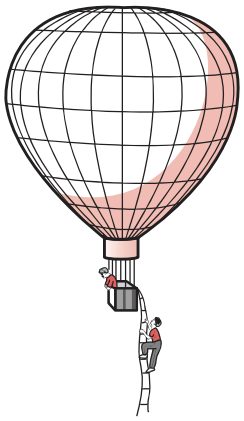

FIGURE 7-31. Problem 1.

2. Two bodies, each made up of weights from a set, are connected by a light cord that passes over a light, frictionless pulley with a diameter of 56.0 mm. The two bodies are at the same level. Each originally has a mass of 850 g. (*a*) Locate their center of mass. (*b*) Thirty-four grams are transferred from one body to the other, but the bodies are prevented from moving. Locate the center of mass. (*c*) The two bodies are now released. Describe the motion of the center of mass and determine its acceleration.

3. A uniform flexible chain of length L, with weight per unit length λ, passes over a small, frictionless peg; see Fig. 7-32. It is released from a rest position with a length of chain x hanging from one side and a length $L - x$ from the other side. Find the acceleration a as a function of x.

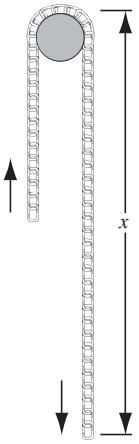

FIGURE 7-32. Problem 3.

4. A cannon and a supply of cannonballs are inside a sealed railroad car of length L, as in Fig. 7-33. The cannon fires to the right; the car recoils to the left. The cannonballs remain in the car after hitting the far wall. (*a*) After all the cannonballs have been fired, what is the greatest distance the car can have moved from its original position? (*b*) What is the speed of the car after all the cannonballs have been fired?

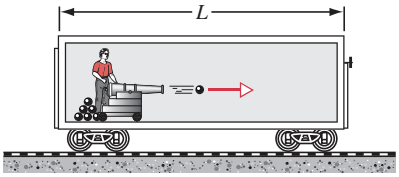

FIGURE 7-33. Problem 4.

5. A cylindrical storage tank is initially filled with aviation gasoline. The tank is then drained through a valve on the bottom. See Fig. 7-34. (*a*) As the gasoline is withdrawn, describe qualitatively the motion of the center of mass of the tank and its remaining contents. (*b*) What is the depth x to which the tank is filled when the center of mass of the tank and its remaining contents reaches its lowest point? Express your answer in terms of H, the height of the tank; M, its mass; and m, the mass of gasoline it can hold.

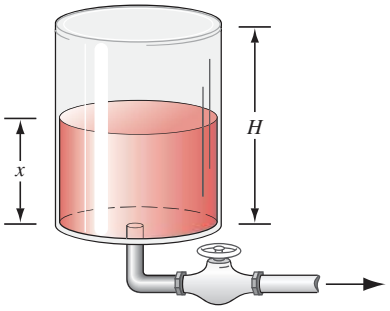

FIGURE 7-34. Problem 5.

6. Find the center of mass of a homogeneous semicircular plate. Let R be the radius of the circle.

7. A 1400-kg cannon, which fires a 70.0-kg shell with a muzzle speed of 556 m/s, is set at an elevation angle of 39.0° above the horizontal. The cannon is mounted on frictionless rails, so that it recoils freely. (*a*) What is the speed of the shell with respect to the Earth? (*b*) At what angle with the ground is the shell projected? (Hint: The horizontal component of the momentum of the system remains unchanged as the gun is fired.)

8. A rocket sled with a mass of 2870 kg moves at 252 m/s on a set of rails. At a certain point, a scoop on the sled dips into a trough of water located between the tracks and scoops water

into an empty tank on the sled. Determine the speed of the sled after 917 kg of water have been scooped up.

9. To keep a conveyor belt moving when it transports luggage requires a greater driving force than for an empty belt. What additional driving force is needed if the belt moves at a constant speed of 1.5 m/s and the rate at which luggage is placed on one end of the belt and removed at the other end is 20 kg/s? Assume that the luggage is dropped vertically onto the belt; persons removing luggage grab hold of it and bring it to rest relative to themselves before lifting it off the belt.

10. A 5860-kg rocket is set for vertical firing. The exhaust speed is 1.17 km/s. How much gas must be ejected each second to supply the thrust needed (*a*) to overcome the weight of the rocket and (*b*) to give the rocket an initial upward acceleration of 18.3 m/s^2? Note that, in contrast to the situation described in Sample Problem 7-9, gravity is present here as an external force.

11. Two long barges are floating in the same direction in still water, one with a speed of 9.65 km/h and the other with a speed of 21.2 km/h. While they are passing each other, coal is shoveled from the slower to the faster one at a rate of 925 kg/min; see Fig. 7-35. How much additional force must be provided by the driving engines of each barge if neither is to change speed? Assume that the shoveling is always perfectly sideways and that the frictional forces between the barges and the water do not depend on the weight of the barges.

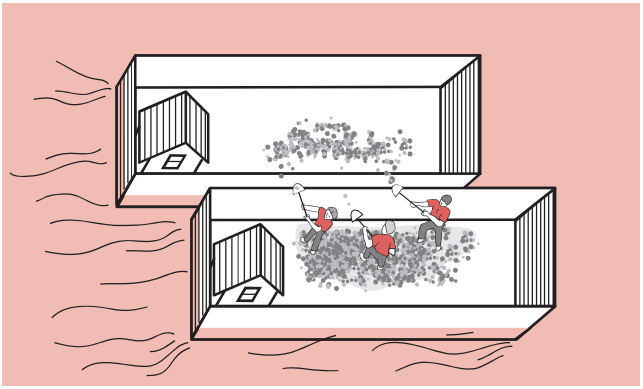

FIGURE 7-35. Problem 11.

12. A flexible inextensible string of length L is threaded into a smooth tube, into which it snugly fits. The tube contains a right-angled bend and is positioned in the vertical plane so that one arm is vertical and the other horizontal. Initially, at $t = 0$, a length y_0 of the string is hanging down in the vertical arm. The string is released and slides through the tube, so that at any subsequent time t later, it is moving with a speed dy/dt, where $y(t)$ is the length of the string that is then hanging vertically. (*a*) Show that in terms of the variable-mass problem $v_{rel} = 0$, so that the equation of motion has the form $m\,dv/dt = F_{ext}$. (*b*) Show that the specific equation of motion is $d^2y/dt^2 = gy/L$. (*c*) Show that

$$y = (y_0/2)(e^{\sqrt{g/L}\,t} + e^{-\sqrt{g/L}\,t})$$

is a solution to the equation of motion [by substitution into (*b*)] and discuss the solution.

COMPUTER PROBLEM

1. A medium-sized rocket has a mass of 4000 kg when empty and can hold 27,000 kg of fuel and oxidizer. The engine burns fuel at a rate of 230 kg/s and the speed of the exhaust gas is a constant 2,500 m/s. Assume the rocket is launched vertically and that there is minimal air friction. Numerically solve for the trajectory of the rocket. Include effects caused by the variation of free-fall acceleration with altitude,

$$g = (9.8\ \text{m/s}^2)\left(\frac{R_E}{R_E + y}\right)^2,$$

where y is the altitude of the rocket above the ground and R_E is the radius of the Earth. (*a*) At what altitude does burn-out occur? (*b*) What is the speed of the rocket at this point? (*c*) What will be the highest point of the trajectory?